HIGHER ALGEBRA

WELLS

A

SHORT COURSE

IN

HIGHER ALGEBRA

FOR

ACADEMIES, HIGH SCHOOLS, AND COLLEGES.

BY

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PREFACE.

THIS volume has been prepared in response to a demand from numerous teachers for a work intermediate in its scope between the author's Academic and University Algebras. The first 248 pages of the Higher Algebra are the same as the corresponding pages of the Academic Algebra. The work on pages 249 to 305 of the latter has been rewritten with reference to the new matter, and 71 pages have been added to the book; and it is now put forth as a complete preparatory text, containing all the topics required for admission to any of the Colleges, Universities, or Scientific Schools of the country.

The new matter is contained principally in the following chapters:

XXVI. Inequalities.

XXVII. The Theory of Limits; Interpretation of the forms $\frac{a}{0}$, $\frac{a}{\infty}$, and $\frac{0}{0}$.

XXIX. Variation.

XXXII. Harmonical Progression.

XXXIV. The Theorem of Undetermined Coefficients.

XXXV. The Binomial Theorem; Fractional and Negative Exponents.

XXXVII. Compound Interest and Annuities.

XXXVIII. Permutations and Combinations.

XXXIX. Continued Fractions.

There is also given in connection with the chapter on Logarithms, a discussion of Logarithmic and Exponential Series.

Attention is respectfully invited to the following among the many new features of the book:

1. The method of factoring quadratic expressions, Art. 283; and the examples in the same article illustrating the factoring of expressions of six terms.

2. The method of interpreting the forms $\frac{a}{0}$ and $\frac{a}{\infty}$; Arts. 300 and 301.

3. The fundamental ideas with regard to convergency and divergency of series; Arts. 371 to 373.

4. The method given in Ex. 2, Art. 381, for finding the coefficients when separating a fraction into its partial fractions.

5. The proof of the Binomial Theorem for any form of exponent, Arts. 389 to 391; especially the general proof, in Art. 390, of the law of formation of the successive coefficients.

6. The proof of the formula for the number of permutations of n quantities taken r at a time; Art. 447.

WEBSTER WELLS.

PREFACE TO EDITION OF 1895.

Since the printing of the first edition of this work, certain topics in the General Theory of Equations have been added to the entrance requirements of several American colleges and scientific schools.

To meet these advanced requirements, the author has added the following chapters to the book:

XL. General Theory of Equations.
XLI. Solution of Higher Equations.

W. W.

Mass. Institute of Technology, 1895.

CONTENTS.

ALGEBRA.

I. DEFINITIONS AND NOTATION.

1. Algebra is that branch of mathematics in which the relations of numbers are investigated, and the reasoning abridged and generalized by means of *symbols*.

Note. Writers on Algebra employ the word "quantity" as synonymous with "number"; this definition of the word will be understood throughout the present work.

2. The **Symbols** of Algebra are of four kinds:

1. Symbols of Quantity.
2. Symbols of Operation.
3. Symbols of Relation.
4. Symbols of Abbreviation.

SYMBOLS OF QUANTITY.

3. The symbols of quantity generally used are the *figures* of Arithmetic, and the *letters* of the alphabet.

Figures are used to represent known quantities and determined values; while letters may represent any quantities whatever, known or unknown.

4. Known Quantities, or those whose values are given, when not expressed by figures, are usually represented by the first letters of the alphabet, as a, b, c.

5. Unknown Quantities, or those whose values are to be determined, are usually represented by the last letters of the alphabet, as x, y, z.

6. Quantities occupying similar relations in the same problem, are often represented by the same letter, distinguished by different *accents;* as a', a'', a''', read "a prime," "a second," "a third," etc.

They may also be distinguished by different *subscript* figures; as a_1, a_2, a_3, read "a one," "a two," "a three," etc.

7. Zero, or the absence of quantity, is represented by the symbol 0.

SYMBOLS OF OPERATION.

8. The **Sign of Addition,** $+$, is called "*plus.*"

Thus, $a + b$, read "a plus b," indicates that the quantity b is to be added to the quantity a.

9. The **Sign of Subtraction,** $-$, is called "*minus.*"

Thus, $a - b$, read "a minus b," indicates that the quantity b is to be subtracted from the quantity a.

Note. The sign $\sim$ indicates the difference of two quantities; thus, $a \sim b$ denotes that the difference of the quantities a and b is to be found.

10. The **Sign of Multiplication,** $\times$, is read "*times,*" "*into,*" or "*multiplied by.*"

Thus, $a \times b$ indicates that the quantity a is to be multiplied by the quantity b.

The sign of multiplication is usually omitted in Algebra, except between arithmetical figures; the multiplication of quantities is therefore indicated by the absence of any sign between them. Thus, $2ab$ indicates the same as $2 \times a \times b$.

A point is sometimes used in place of the sign $\times$ between two or more figures; thus, $2 \cdot 3 \cdot 4$ denotes $2 \times 3 \times 4$.

11. Quantities multiplied together are called *factors*, and the result of the multiplication is called the *product*.

Thus, 2, a, and b are the factors of the product $2ab$.

12. A **Coefficient** is a number prefixed to a quantity to indicate how many times the quantity is to be taken.

Thus, in $4ax$, 4 is the coefficient of ax, and indicates that ax is to be taken 4 times; that is, $4ax$ is equivalent to $ax + ax + ax + ax$.

When no coefficient is expressed, 1 is understood to be the coefficient. Thus, a is the same as $1a$.

When any number of factors are multiplied together, the product of any of them may be regarded as the coefficient of the product of the others. Thus, in $abcd$, ab is the coefficient of cd; b of acd; abd of c; etc.

13. An **Exponent** is a figure or letter written at the right of, and above a quantity, to indicate the number of times the quantity is taken as a factor.

Thus, in x^3, the ^{3} indicates that x is taken three times as a factor; that is, x^3 is equivalent to xxx.

14. The product obtained by taking a factor two or more times is called a *power*. A single letter is also often called the *first* power of that letter. Thus,

a^2 is read "a to the second power," or "a *square*," and indicates aa;

a^3 is read "a to the third power," or "a *cube*," and indicates aaa;

a^4 is read "a to the fourth power," or "a *fourth*," and indicates $aaaa$; etc.

When no exponent is written, the *first* power is understood; thus, a is the same as a^1.

15. The **Sign of Division**, $\div$, is read "*divided by*."

Thus, $a \div b$ denotes that the quantity a is to be divided by the quantity b.

Division is also indicated by writing the dividend above, and the divisor below, a horizontal line. Thus, $\frac{a}{b}$ indicates the same as $a \div b$. When thus written, $\frac{a}{b}$ is often read "a *over* b."

SYMBOLS OF RELATION.

16. The symbols of relation are signs used to indicate the relative magnitudes of quantities.

17. The **Sign of Equality,** $=$, read "*equals,*" or "*is equal to,*" indicates that the quantities between which it is placed are equal.

Thus, $x = y$ indicates that the quantities x and y are equal.

A statement that two quantities are equal is called an *equation.*

Thus, $x + 4 = 2x - 1$ is an equation, and is read "x plus 4 equals $2x$ minus 1."

18. The **Sign of Inequality,** $>$ or $<$, read "*is greater than*" and "*is less than*" respectively, when placed between two quantities, indicates that the quantity toward which the opening of the sign turns is the greater.

Thus, $x > y$ is read "x is greater than y"; $x - 6 < y$ is read "x minus 6 is less than y."

SYMBOLS OF ABBREVIATION.

19. The **Sign of Deduction,** $\therefore$, stands for *therefore* or *hence.*

20. The **Signs of Aggregation,** the *parenthesis* ( ), the *brackets* [ ], the *braces* { }, and the *vinculum* ——, indicate that the quantities enclosed by them are to be taken collectively. Thus,

$$(a+b)x,\quad [a+b]x,\quad \{a+b\}x,\quad \overline{a+b}\times x,$$

all indicate that the quantity obtained by adding a and b is to be multiplied by x.

21. The **Sign of Continuation**, ..., stands for "*and so on*" or "*continued by the same law.*" Thus,

$$a,\ a+b,\ a+2b,\ a+3b,\ \ldots$$

reads "a, a plus b, a plus $2b$, a plus $3b$, and so on."

ALGEBRAIC EXPRESSIONS.

22. An **Algebraic Expression** is any combination of algebraic symbols; as $2x^2-3ab+c^3$.

23. A **Term** is an algebraic expression whose parts are not separated by the signs $+$ or $-$; as $2x^2$, $-3ab$, or $+c^3$.

$2x^2$, $-3ab$, and c^3 are called the terms of the expression $2x^2-3ab+c^3$.

24. **Positive Terms** are those preceded by a *plus* sign; as

$$+2x^2, \text{ or } +c^3.$$

For this reason, the sign $+$ is often called the *positive sign*. If no sign is expressed, the term is understood to be positive; thus, a is the same as $+a$.

25. **Negative Terms** are those preceded by a *minus* sign; as

$$-3ab, \text{ or } -bc^2.$$

For this reason, the sign $-$ is often called the *negative sign;* it can never be omitted before a negative term.

Note. In a negative term, the numerical coefficient indicates how many times the quantity is to be taken *subtractively*. (Compare Art. 12.)

Thus, $-3ab$ is equivalent to $-ab-ab-ab$.

26. In Arithmetic, if the same number be both added to and subtracted from another, the value of the latter will not be changed. Thus,

$$5+3-3=5.$$

Similarly, in Algebra, if any quantity b be both added to and subtracted from another quantity a, the result will be equal to a. That is,

$$a+b-b=a.$$

Consequently, equal terms affected by *unlike signs*, in an expression, neutralize each other, or *cancel*.

27. A **Monomial** is an algebraic expression consisting of only one term; as $5a$, $7ab$, or $-3b^2c$.

A monomial is sometimes called a *simple* quantity.

28. A **Polynomial** is an algebraic expression consisting of more than one term; as $a+b$, or $3a^2+b-5b^3$.

A polynomial is sometimes called a *compound* quantity.

29. A **Binomial** is a polynomial of two terms; as $a-b$, or $2a+b^2$.

30. A **Trinomial** is a polynomial of three terms; as $ab+2c^2-b^3$.

31. **Similar** or **Like Terms** are those which differ only in their numerical coefficients. Thus,

$2xy^2$ and $-7xy^2$ are similar terms.

32. **Dissimilar** or **Unlike Terms** are those which are not similar. Thus,

bx^2y and bxy^2 are dissimilar terms.

33. The **Degree** of a term is the number of *literal* factors which it contains. Thus,

$2a$ is of the *first* degree, since it contains but *one* literal factor;

ab is of the *second* degree, since it contains *two* literal factors;

$3a^2b^3$ is of the *fifth* degree, since it contains *five* literal factors.

The degree of any term is determined by adding the exponents of its several letters. Thus,

ab^2c^3 is of the *sixth* degree.

34. Homogeneous Terms are those of the same degree. Thus,

a^2, $3bc$, and $-4x^2$ are homogeneous terms.

35. A polynomial is called homogeneous when all its terms are homogeneous; as $a^3 + 2abc - 3b^3$.

36. A polynomial is said to be *arranged* according to the *ascending* powers of any letter, when the term containing the lowest exponent of that letter is placed first, that having the next higher immediately after, and so on. Thus,

$$b^4 + 3ab^3 - 2a^2b^2 + 3a^3b - 4a^4$$

is arranged according to the ascending powers of a.

Note. The term b^4, which does not involve a at all, is regarded as containing the lowest exponent of a in the above expression.

37. A polynomial is said to be arranged according to the *descending* powers of any letter, when the term containing the highest exponent of that letter is placed first, that having the next lower immediately after, and so on. Thus,

$$b^4 + 3ab^3 - 2a^2b^2 + 3a^3b - 4a^4$$

is arranged according to the descending powers of b.

38. The **Reciprocal** of a quantity is 1 divided by that quantity. Thus, the reciprocal of a is $\frac{1}{a}$; and of $x+y$ is $\frac{1}{x+y}$.

39. The **Numerical Value** of an expression is the result obtained by rendering it into an arithmetical quantity, by means of the numerical values assigned to its letters.

Thus, the numerical value of

$$4a + 3bc - d^3$$

when $a = 4$, $b = 3$, $c = 5$, and $d = 2$, is

$$4 \times 4 + 3 \times 3 \times 5 - 2^3 = 16 + 45 - 8 = 53.$$

EXERCISES.

40. Find the numerical value of the following expressions, when $a = 2$, $b = 3$, $c = 1$, and $d = 4$:

1. $a^2 + 2ab - c + d.$
2. $3a^3 - 2a^2b + c^3.$
3. $5a^2b + 4ab^2 - 27cd.$
4. $2a^2 + 3bc - \dfrac{5}{cd}.$
5. $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d}.$
6. $\dfrac{b^a}{a^d}.$
7. $\dfrac{cd}{b^2} + \dfrac{ab}{c^2}.$
8. $b^d - a^b b^2.$
9. $\dfrac{3d^2}{5ac} - \dfrac{2a}{3b^2}.$
10. $\dfrac{a^2}{b^2} + \dfrac{b^2}{c^2} + \dfrac{c^2}{d^2}.$

If the expression involves parentheses, the operations indicated *within* the parentheses must be performed *first*.

Thus, to find the numerical value, when $a = 3$, $b = 2$, and $c = 1$, of

$$b(2a - 3c)(a^2 + c^2) - \frac{a^2 + b^2}{a^2 - b^2},$$

we have,

$$\begin{aligned} 2a - 3c &= 6 - 3 = 3 \\ a^2 + c^2 &= 9 + 1 = 10 \\ a^2 + b^2 &= 9 + 4 = 13 \\ a^2 - b^2 &= 9 - 4 = 5 \end{aligned}$$

Hence the numerical value of the expression is

$$2 \times 3 \times 10 - \frac{13}{5} = 60 - \frac{13}{5} = \frac{287}{5}.$$

Find the numerical values of the following, when $a=4$, $b=2$, $c=3$, and $d=1$:

11. $a^2(a+b)-2abc.$

12. $7a^2+(a-b)(a-c).$

13. $15a-7(b+c)-d.$

14. $c(a^{b+d}+a^{b-d}).$

15. $25a^2-7(b^2+c^2)+d^2.$

16. $\dfrac{4}{3a-3c}+\dfrac{8}{3}.$

17. $\dfrac{25a-30c-d}{b+c}.$

18. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{c^2-b^2}{c^2+b^2}.$

Find the values of the following, when $a=\frac{1}{2}$, $b=\frac{1}{3}$, $c=\frac{1}{5}$, and $x=2$:

19. $(2a+3b+5c)(8a+3b-5c)(2a-3b+15c).$

20. $x^3+\left(\dfrac{1}{a}+\dfrac{1}{b}\right)x^2+\left(\dfrac{1}{b}-\dfrac{1}{a}\right)x+\dfrac{2}{b^2}.$

21. $x^4-(2a+3b)x^3+(3a-2b)x^2-cx+bc.$

22. $\dfrac{a^2-\dfrac{b}{2}}{8bc-a}-\dfrac{x}{a+b+c}.$

41. Put the following into the form of algebraic expressions:

1. Five times a, added to twice b.

2. Two times x, minus y to the second power.

3. The product of a, b, c square, and d cube.

4. Three times the cube of a, minus twice the product of a square and b, plus the cube of c.

5. The product of $x+y$ and a.

6. The product of $x+y$ and $a-b$.

7. a square, divided by the product of b and c.

8. a square, divided by $b-c$.

9. x divided by 3, plus 2, equals three times y minus 11.

10. The product of m and $a+b$ is less than the reciprocal of x cube.

AXIOMS.

42. An **Axiom** is a truth assumed as self-evident.

Algebraic operations are based upon definitions, and the following axioms:

1. If equal quantities be added to equal quantities, the sums will be equal.
2. If equal quantities be subtracted from equal quantities, the remainders will be equal.
3. If equal quantities be multiplied by equal quantities, the products will be equal.
4. If equal quantities be divided by equal quantities, the quotients will be equal.
5. If the same quantity be both added to and subtracted from another, the value of the latter will not be changed.
6. If a quantity be both multiplied and divided by another, the value of the former will not be changed.
7. Quantities which are equal to the same quantity are equal to each other.

SOLUTION OF PROBLEMS BY ALGEBRAIC METHODS.

43. The following examples will illustrate the application of the notation of Algebra in the solution of problems.

1. The sum of two numbers is 30, and the greater is 4 times the less. What are the numbers?

We will first solve the problem by the method of Arithmetic, and afterwards by Algebra. The marginal letters refer to the corresponding steps of the two methods; that is, the operation (a) in the algebraic solution is equivalent to the operation (a) in the arithmetical; and so on. In this way the student can compare the two processes step by step.

Solution by Arithmetic.

The less number, plus the greater number, equals 30.

(*a*) Hence the less number, plus 4 times the less number, equals 30

(*b*) Therefore 5 times the less number equals 30.

(*c*) Hence the less number is one-fifth of 30, or 6.

(*d*) Then the greater number is 4 times 6, or 24.

Solution by Algebra.

Let $x =$ the less number.

Then $4x =$ the greater number.

(*a*) By the conditions, $x + 4x = 30.$

(*b*) Or, $5x = 30.$

(*c*) Dividing by 5, $x = 6$, the less number.

(*d*) Whence, $4x = 24$, the greater number.

2. A, B, and C together have $66. A has one-half as much as B, and C has as much as A and B together. How much has each?

Let $x =$ the number of dollars A has.

Then $2x =$ the number of dollars B has,

and $x + 2x$, or $3x =$ the number of dollars C has.

By the conditions, $x + 2x + 3x = 66.$

Or, $6x = 66.$

Whence, $x = 11$, the number of dollars A has,

Therefore, $2x = 22$, the number of dollars B has,

and $3x = 33$, the number of dollars C has.

3. The sum of the ages of A and B is 109 years, and A is 13 years younger than B. What are their ages?

Let $x =$ the number of years in A's age.

Then $x + 13 =$ the number of years in B's age.

By the conditions, $x + x + 13 = 109.$

Or, $2x + 13 = 109.$

Whence, $2x = 96.$

And, $x = 48$, the number of years in A's age

Therefore, $x + 13 = 61$, the number of years in B's age

PROBLEMS.

4. The greater of two numbers is 5 times the less, and their sum is 42. What are the numbers?

5. The sum of the ages of A and B is 68 years, and B is 6 years older than A. What are their ages?

6. Divide $1200 between A and B, so that A may receive $128 less than B.

7. A man had $3.72; after spending a certain sum, he found that he had left 3 times as much as he had spent. How much had he spent?

8. Divide $260 between A, B, and C, so that B may receive 3 times as much as A, and C 3 times as much as B.

9. Divide the number 125 into two parts, one of which is 21 less than the other.

10. The sum of three numbers is 98; the second is 3 times the first, and the third exceeds the second by 7 What are the numbers?

11. A, B, and C together have $127; C has twice as much as A, and $13 more than B. How much has each?

12. My horse, carriage, and harness together are worth $400. The horse is worth 11 times as much as the harness, and the carriage is worth $175 less than the horse. What is the value of each?

13. The sum of three numbers is 108. The first is one-third of the second, and 33 less than the third. What are the numbers?

14. Divide the number 210 into three parts, such that the first is one-half of the second, and one-third of the third.

15. A man bought a cow, a sheep, and a hog, for $75; the price of the sheep was $27 less than the price of the cow, and $6 more than the price of the hog. What was the price of each?

NEGATIVE QUANTITIES.

44. The signs + and −, besides indicating the operations of addition and subtraction, are also used, in Algebra, to distinguish between quantities which are the exact reverse of each other in quality or condition.

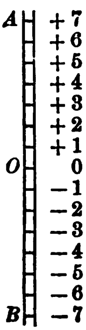
Thus, in the thermometer, we may speak of a temperature above zero as +, and of one below as −. For example, $+25°$ means 25° above zero, and $-10°$ means 10° below zero.

In navigation, north latitude is considered +, and south latitude −; west longitude is considered +, and east longitude −. For example, a place in latitude $-30°$, longitude $+95°$, would be in latitude 30° south of the equator, and in longitude 95° west of Greenwich.

Again, in financial transactions, we may consider assets as +, and debts or liabilities as −. For example, the statement that a man's property is $-\$100$, means that he owes or is in debt \$100.

And in general, when we have to consider quantities the exact reverse of each other in quality or condition, we may regard quantities of either quality or condition as positive, and those of the opposite quality or condition as negative.

45. The thermometer affords an excellent illustration of the relation between positive and negative quantities.



Let OA represent the scale for temperatures above zero, and OB the scale for temperatures below zero; and let us consider the following problem:

At 7 A.M. the temperature is $-6°$; at noon it is 11° warmer, and at 6 P.M. it is 9° colder than at noon. Required the temperatures at noon and at 6 P.M.

Beginning at the scale-mark −6, and counting

11 degree-spaces upwards, we reach the scale-mark $+5$; and counting from the latter 9 degree-spaces downwards, we reach the scale-mark -4. Hence, the temperature at noon is $+5°$, and at 6 P.M. $-4°$.

EXERCISES.

46. 1. At 7 A.M. the temperature is $-8°$; at noon it is 7° warmer, and at 6 P.M. it is 3° colder than at noon. Required the temperatures at noon and at 6 P.M.

2. A certain city was founded in the year 151 B.C., and was destroyed 203 years later. In what year was it destroyed?

3. At 7 A.M. the temperature is $+4°$; at noon it is 10° colder, and at 6 P.M. it is 6° warmer than at noon. Required the temperatures at noon and at 6 P.M.

4. What is the difference in latitude between two places whose latitudes are $+56°$ and $-31°$?

5. A man has bills receivable to the amount of $2000, and bills payable to the amount of $3000. How much is he worth?

6. At 7 A.M. the temperature is $-3°$, and at noon it is $+11°$. How many degrees warmer is it at noon than at 7 A.M.?

7. What is the difference in longitude between two places whose longitudes are $+25°$ and $-90°$?

8. The temperature at 6 A.M. is $-7°$, and during the morning it grows warmer at the rate of 3° an hour. Required the temperatures at 8 A.M., at 9 A.M., and at noon.

47. The *absolute value* of a quantity is the number represented by the quantity, taken independently of the sign affecting it.

Thus, the absolute value of -5 is 5.

II. ADDITION.

48. Addition, in Algebra, is the process of collecting two or more quantities into one equivalent expression, called the *sum*.

Thus, the sum of a and b is $a+b$ (Art. 8).

49. If either quantity is negative, or a polynomial, it should be enclosed in a parenthesis (Art. 20); thus,

The sum of a and $-b$ is indicated by $a+(-b)$.

The sum of $a-b$ and $c-d$ is indicated by

$$(a-b)+(c-d).$$

50. Required the sum of a and $-b$.

Using the interpretation of negative quantities as explained in Art. 44, if a man incurs a debt of $100, we may regard the transaction either as adding −$100 to his property, or as subtracting $100 from it That is,

Adding a negative quantity is equivalent to subtracting a positive quantity of the same absolute value (Art. 47).

Thus, the sum of a and $-b$ is obtained by subtracting b from a; or,

$$a+(-b)=a-b.$$

51. It follows from Arts. 48 and 50 that the addition of monomials is effected by *uniting the quantities with their respective signs.*

Thus, the sum of a, $-b$, c, $-d$, and $-e$, is

$$a-b+c-d-e.$$

It is immaterial in what order the terms are united, provided each has its proper sign. Thus, the above result may also be expressed

$$c+a-e-d-b,$$

$$-d-b+c-e+a, \text{ etc.}$$

ADDITION OF SIMILAR TERMS.

52. 1. Required the sum of $5a$ and $3a$.

$5a$ signifies a taken 5 times (Art. 12), and $3a$ signifies a taken 3 times. We have, therefore, a taken in all 8 times, or $8a$. That is,

$$5a + 3a = 8a.$$

2. Required the sum of $-5a$ and $-3a$.

$-5a$ signifies a taken 5 times *subtractively* (Art. 25), and $-3a$ signifies a taken 3 times subtractively. We have, therefore, a taken in all 8 times subtractively, or $-8a$. That is,

$$-5a - 3a = -8a.$$

Therefore,

To add two similar (Art. 31) *terms of like sign, add the coefficients, affix to the result the common symbols, and prefix the common sign.*

53. 1. Required the sum of $8a$ and $-5a$.

Since $8a$ is the sum of $3a$ and $5a$ (Art. 52, 1), the sum of $8a$ and $-5a$ is equal to the sum of $3a$, $5a$, and $-5a$, which is

$$3a + 5a - 5a. \qquad \text{(Art. 51.)}$$

But, by Art. 26, $5a$ and $-5a$ cancel each other, leaving the result $3a$.

Hence, $$8a + (-5a) = 3a.$$

2. Required the sum of $-8a$ and $5a$.

Since $-8a$ is the sum of $-3a$ and $-5a$ (Art. 52, 2), the sum of $-8a$ and $5a$ is equal to the sum of $-3a$, $-5a$, and $5a$, which is

$$-3a - 5a + 5a, \text{ or } -3a.$$

Hence, $$(-8a) + 5a = -3a.$$

Therefore,

To add two similar terms of unlike sign, subtract the less coefficient from the greater, affix to the result the common symbols, and prefix the sign of the greater coefficient.

Note. A clear understanding of the nature of the processes in Arts. 52 and 53 may be obtained by comparing them with the following, the negative quantities being interpreted as explained in Art. 44.

1. If a man owes \$5, and incurs a debt of \$3, he will be in debt to the amount of \$8. That is, the sum of $-\$5$ and $-\$3$ is $-\$8$.

2. If a man's assets amount to \$8, and his liabilities to \$5, he is worth \$3. That is, the sum of \$8 and $-\$5$ is \$3.

3. If a man's liabilities amount to \$8, and his assets to \$5, he is in debt to the amount of \$3. That is, the sum of $-\$8$ and \$5 is $-\$3$.

EXAMPLES.

54. Add the following:

1. 11 and -5.
2. -13 and 3.
3. 12 and -1.
4. -4 and -7.
5. $-2a$ and $7a$.
6. b and $-3b$.
7. $-11m$ and $-8m$.
8. bc and $16bc$.
9. $-2ax$ and $7ax$.
10. $-3a^2b^2$ and $-a^2b^2$.
11. $12mn^2$ and $-19mn^2$.
12. $-13abc$ and $22abc$.

13. Required the sum of $2a$, $-a$, $3a$, $-12a$, and $6a$.

Since the order of the terms is immaterial (Art. 51), we may add the positive terms first, and then the negative, and finally combine these results by the rule of Art. 53.

The sum of $2a$, $3a$, and $6a$ is $11a$.
The sum of $-a$ and $-12a$ is $-13a$.

Hence, the required sum is $11a - 13a$, or $-2a$. *Ans.*

Add the following:

14. $7a$, $-a$, and $-3a$.
15. $-6m$, m, $-11m$, and $5m$.
16. $13ab$, $-7ab$, $-8ab$, and $-6ab$.

17. $7n^2$, $-n^2$, $-3n^2$, $11n^2$, and $-10n^2$.

18. $13ax^3$, $-ax^3$, $-20ax^3$, $6ax^3$, and $-5ax^3$.

If the terms are not all similar, we may combine the similar terms, and unite the others with their respective signs.

19. Required the sum of $12a$, $-5x$, $-3y$, $-5a$, $8x$, and $-3x$.

The sum of $12a$ and $-5a$ is $7a$.
The sum of $-5x$, $8x$, and $-3x$ is 0.

Hence, the required sum is $7a-3y$. *Ans.*

Add the following:

20. $5ax$, $-11b$, $-ax$, and $6b$.

21. $2a$, $5b$, $-3c$, $-8b$, and $9c$.

22. $5m$, $-2n^2$, n, $-2m$, $-b^2$, and $3n^2$.

23. $3x$, $-y$, $-x$, 6, $-8y$, $-2x$, $4y$, and -5.

ADDITION OF POLYNOMIALS.

55. A polynomial may be regarded as the sum of its monomial terms (Art. 51). Thus, $2a-3b+4c$ is the sum of the terms $2a$, $-3b$, and $4c$.

Hence, the addition of two or more polynomials is effected by *uniting their terms with their respective signs.*

Thus, the sum of $a-b$ and $c-d$ is $a-b+c-d$.

56. Required the sum of $6a-7x$, $3x-2a+3y$, and $2x-a-mn$.

It is convenient in practice to set the expressions down one underneath the other, similar terms being in the same vertical column. Thus,

$$\begin{array}{rrrr}
6a & -7x & & \\
-2a & +3x & +3y & \\
-\ \ a & +2x & & -mn \\
\hline
3a & -2x & +3y & -mn,
\end{array}\ \textit{Ans.}$$

From the above principles we derive the following rule:

To add two or more expressions, set them down one underneath the other, similar terms being in the same vertical column. Find the sum of the terms in each column, and unite the results with their respective signs.

EXAMPLES.

57. Add the following:

1.	2.	3.
$2a-7x$	$-3ab+2cd$	$-11a-5mp^2$
$-a+4x$	$-7ab+8cd$	$8a+11mp^2$
$a+x$	$4ab-6cd$	$-9a-7mp^2$

4. $2a-3b+5c$ and $b-5c+2d$.

5. $9mn^2+x^2y$, $-mn^2+3x^2y$, and $-6mn^2-7x^2y$.

6. $a^2-2ab+b^2$, $a^2+2ab+b^2$, and $2a^2-2b^2$.

7. $3a^2+2ab+4b^2$, $5a^2-8ab+b^2$, and $-6a^2+5ab-5b^2$.

8. $6x^3-7x-4$, x^2-x-2, and $8x-9x^2-x^3$.

9. $4mn+3ab-4c$, $3x-4ab+2mn$, and $3m^2-4x$.

10. $3x-2y-z$, $6y-5x-7z$, $8z-y-x$, and $4x-9y$.

11. $6x-3y+7m$, $2n-x+y$, $2y-4x-5m-9n$, and $m-2x$.

12. $2x^3-5x^2-x+7$, $3x^2-2-6x^3+8x$, $x+3x^3-4$, and $1+2x^2-5x$.

13. $2a-3b+4d$, $2b-3d+4c$, $2d-3c+4a+4b$, and $2c-3a$.

14. $2a^3-a^2b-2b^3$, $8a^3-8ab^2-3b^3$, $3a^2b-ab^2+b^3$, and $6ab^2-2a^2b-5a^3$.

15. $4x^3-10a^3-5ax^2+6a^2x$, $6a^3+3x^3+4ax^2+2a^2x$, $-17x^3+19ax^2-15a^2x$, and $6x^3+7a^2x+5a^3-18ax^2$.

III. SUBTRACTION.

58. Subtraction, in Algebra, is the process of taking one quantity from another.

The *Subtrahend* is the quantity to be subtracted.

The *Minuend* is the quantity from which it is to be subtracted.

The *Remainder* is the result of the operation.

59. It is evident from the above that the minuend is equal to the sum of the subtrahend and the remainder.

60. Let it be required to subtract $-b$ from a.

Using the interpretation of negative quantities as explained in Art. 44, if a man cancels a debt of $100, we may regard the transaction either as subtracting −$100 from his property, or as adding $100 to it. That is,

Subtracting a negative quantity is equivalent to adding a positive quantity of the same absolute value.

Thus, to subtract $-b$ from a, we add b to a; or

$$a-(-b)=a+b.$$

Hence, *to subtract one quantity from another, change the sign of the subtrahend, and add the result to the minuend.*

61. 1. Subtract $5a$ from $2a$.

By Art. 60, the result is equal to the sum of $-5a$ and $2a$, which is $-3a$.

2. From $-2a$ subtract $5a$.

The result is equal to the sum of $-2a$ and $-5a$, or $-7a$.

3. From $5a$ take $-2a$.

Result, $5a+2a$, or $7a$.

4. From $-2a$ take $-5a$.

Result, $-2a+5a$, or $3a$.

EXAMPLES.

62. Subtract the following:

1. -3 from 11.	**3.** -8 from -3.	**5.** 23 from 10.
2. 16 from -5.	**4.** -11 from -17.	**6.** -13 from 11.

7.	**8.**	**9.**	**10.**	**11.**
$27a$	$17x$	$-13y$	$-10mn$	$5a^2b$
$13a$	$-11x$	$4y$	$-18mn$	$14a^2b$

12. From $9ab$ take $-2ab$.

13. From xy take $-cd$.

14. From $17m^3$ take $41m^3$.

15. From $-5x$ take 3.

16. From $-x^2y^2$ take $5x^2y^2$.

17. From $-70abc$ take $-52abc$.

18. From $-7m^2$ take $-8n^2$.

19. From $-33x^3y^2$ take $19x^3y^2$.

20. From $5ab$ take the sum of $9ab$ and $-2ab$.

21. From the sum of $-11x^3$ and $8x^3$ take the sum of $-10x^3$ and $4x^3$.

SUBTRACTION OF POLYNOMIALS.

63. When the subtrahend is a polynomial, each of its terms is to be subtracted from the minuend. Hence,

To subtract one polynomial from another, change the sign of each term of the subtrahend, and add the result to the minuend.

It will be found convenient to place the subtrahend under the minuend, similar terms being in the same vertical column.

64. 1. Subtract $5x^2y-3ab+m^2$ from $3x^2y-2ab+4n$.

Changing the sign of each term of the subtrahend, and adding the result to the minuend, we have

$$\begin{array}{rrrr} 3x^2y & -2ab & +4n & \\ -5x^2y & +3ab & & -m^2 \\ \hline -2x^2y & +ab & +4n & -m^2, \end{array} \quad \textit{Ans.}$$

Note. The student should endeavor to perform *mentally* the operation of changing the sign of each term in the subtrahend, as shown in the following example:

2. From $5a^3 - 7b^3 - 2a^2b$ subtract $3a^2b - 4ab^2 - 2b^3 + a^3$.

$$\begin{array}{r} 5a^3 - 2a^2b \qquad\qquad - 7b^3 \\ a^3 + 3a^2b - 4ab^2 - 2b^3 \\ \hline 4a^3 - 5a^2b + 4ab^2 - 5b^3, \end{array} \textit{ Ans.}$$

EXAMPLES.

Subtract the following:

3.

$$\begin{array}{r} ab + \quad cd - \quad ax \\ 4ab - 3cd - 4ax \\ \hline \end{array}$$

4.

$$\begin{array}{r} 7x + 5y - 3a \qquad \\ x - 7y + 5a - 4 \\ \hline \end{array}$$

5. From $a - b + c$ take $a + b - c$.

6. From $a^2 + 2ab + b^2$ take $a^2 - 2ab + b^2$.

7. From $7abc - 11x + 5y - 48$ take $11abc + 3x + 7y + 100$.

8. Subtract $3m + y^2 - 5a - 7$ from $5m - 3y^2 + 7a - 6$.

9. Subtract $17x^2 + 5y^2 - 4ab + 7$ from $31x^2 - 3y^2 + ab$.

10. From $6a + 3b - 5c + 1$ subtract $6a - 3b - 5c$.

11. From $3m - 5n + r - 2s$ take $2r + 3n - m - 5s$.

12. Take $4a - b + 2c - 5d$ from $d - 3b + a - c$.

13. From $m^2 + 3n^3$ subtract $-4m^2 - 6n^3 + 71x$.

14. From $4c - 3b - 5d + 2x$ take $3a + 8d - b - 6c$.

15. From $a - b - c$ take the sum of $-2a + b + c$ and $a - b + c$.

16. From $x^4 + 2x^3 - 3x + 4$ take $3x^3 + 3x^2 + 5x - 7$.

17. From $4a^3 - 3ab^2 - 5b^3$ subtract $6a^2b - ab^2 + 4b^3$.

18. From $a^2 - 8 + 2a^4 - 3a^3$ take $6a - 11 - 5a^2 - 2a^4$.

19. Take $2x^2 - y^2$ from the sum of $x^2 - 2xy + 3y^2$ and $xy - 4y^2$.

20. From the sum of $x + 2y - 3z$ and $3y - 4x + z$ take $z - 5x + 5y$.

21. From $7a^3 + 3 - 5a^4 + a - 5a^2$ subtract $2a - 6a^2 - 2a^3 + 9 - 11a^4$.

22. From $-7y^3 + 3x^2y - 2x^3 + 6xy^2$ subtract $8x^2y - 2xy^2 + x^3 - 9y^3$.

23. From the sum of $2x^3 - x + 5$ and $x^2 + 8x - 11$ take the sum of $x^3 - 9x^2 - 11x$ and $-4x^3 + 3x^2 - 6$.

24. From the sum of $a^2 + ab + b^2$ and $a^2 - 4ab + 5b^2$ take the sum of $4a^2 + 7b^2 - 2ab$ and $3ab - a^2 - 2b^2$.

25. From $3x^2 - 7y - 2 + xy - 5y^2$ subtraot $-5xy + 6x - 2x^2 - 8 + 2y^2$.

26. From $3x^5 - 8x^4 + 3x^3 - 5x^2 - 2x$ subtract $-3x^4 + 4x^3 + 6x^2 - 6x + 2$.

27. From the sum of $2x^3 - x^2y - 5xy^2$ and $3x^2y - 5xy^2 - 4y^3$ take the sum of $-2x^3 - 7x^2y - 6y^3$ and $-6xy^2 + 5y^3$.

28. From the sum of $a^4 - 1$ and $2a^3 - 10a^2 - 7a$ subtract the sum of $-3a^4 + 2a^2 - 5a$ and $-5a^3 - 12a^2 + 3$.

Note. In Arithmetic, addition always implies *augmentation*, and subtraction *diminution*. In Algebra this is not always the case; for example, in adding -2 to 5, the sum is 3, which is less than 5. Again, in subtracting -2 from 5, the remainder is 7, which is greater than 5.

Thus the terms *Addition, Subtraction, Sum,* and *Remainder* have a much more general signification in Algebra than in Arithmetic.

IV. USE OF PARENTHESES.

65. The use of parentheses (Art. 20) is very frequent in Algebra, and it is necessary to have rules for their removal or introduction.

66. The expression

$$2a-3b+(5b-c+2d)$$

indicates that the quantity $5b-c+2d$ is to be added to $2a-3b$. If the addition be performed, we obtain (Art. 55)

$$2a-3b+5b-c+2d.$$

Again, the expression

$$2a-3b-(5b-c+2d)$$

indicates that the quantity $5b-c+2d$ is to be subtracted from $2a-3b$. If the subtraction be performed, we obtain (Art. 63)

$$2a-3b-5b+c-2d.$$

67. It will be observed that in the first case the signs of the terms within the parenthesis are *unchanged* when the parenthesis is removed; while in the second case the sign of each term within is *changed*, from $+$ to $-$, or from $-$ to $+$.

We have then the following rule for removing a parenthesis:

A parenthesis preceded by a $+$ sign may be removed without altering the signs of the enclosed terms.

A parenthesis preceded by a $-$ sign may be removed, if the sign of each enclosed term be changed, from $+$ to $-$, or from $-$ to $+$.

68. Since the brackets, the braces, and the vinculum (Art. 20) have the same signification as the parenthesis, the rule for their removal is the same.

It should be observed in the case of the vinculum that the sign apparently prefixed to the first term underneath, is in reality the sign of the vinculum. Thus, $+\overline{a-b}$ and $-\overline{a-b}$ are equivalent to $+(a-b)$ and $-(a-b)$, respectively.

EXAMPLES.

69. **1.** Remove the parentheses from

$$2a-3b-(5a-4b)+(4a-b).$$

By the rule of Art. 67, the expression becomes

$$2a-3b-5a+4b+4a-b=a, \textit{ Ans.}$$

Parentheses are often found enclosing others. In this case they may be removed in succession by the rule of Art. 67, and it is better to remove first the *innermost* pair.

2. Simplify the expression

$$4x-\{3x+(-2x-\overline{x-a})\}.$$

We remove the vinculum first, and the others in succession. Thus,

$$\begin{aligned} &4x-\{3x+(-2x-\overline{x-a})\} \\ =&4x-\{3x+(-2x-x+a)\} \\ =&4x-\{3x-2x-x+a\} \\ =&4x-3x+2x+x-a=4x-a, \textit{ Ans.} \end{aligned}$$

Reduce the following expressions to their simplest forms by removing the parentheses, etc., and uniting similar terms:

3. $a-(b-c)+(-d+e)$.

4. $5x-\{2x-3y\}-[-2x+4y]$.

5. $a-b+c-\overline{a+b-c}-\overline{c-b-a}$.

6. $m^2-2n+\{a-n+3m^2\}-\overline{5a+3n-m^2}$.

7. $a^2-b^2-(a^2-2ab+b^2)-[a^2+2ab+b^2]$.

8. $3a-(2a-\{a+2\})$.

9. $a-(b+\{-c+d\}-e)$.

10. $a-[(-b+c)-(d-e)]$.

11. $3x-[2y+\overline{x-y}]+[3y-\overline{2x+y}]$.

12. $14x-(5x-9)-\{4-3x-(2x-3)\}$.

13. $2m-[n-\{3m-(2n-m)\}]$.

14. $3x-(5x+[-4x-\overline{y-x}])-(-x-3y)$.

15. $3c+(2a-[5c-\{3a+\overline{c-4a}\}])$.

16. $5a-(4a-\{-3a-[2a-\overline{a-1}]\})$.

17. $8x-[5x-(3x-4)-\{7x+(-9x+2)\}]$.

18. $2m-[3m-\{m-(2m-\overline{3m+4})\}-(5m-2)]$.

19. $c-[2c-(6a-b)-\{c-(5a+2b)-(a-3b)\}]$.

20. $3a-\{b-[b-(a+b)-\{-b-(b-\overline{a-b})\}]\}$.

70. To enclose any number of terms in a parenthesis, we take the converse of the rule of Art. 67:

Any number of terms may be enclosed in a parenthesis preceded by a + sign, without altering their signs.

Any number of terms may be enclosed in a parenthesis preceded by a − sign, if the sign of each term be changed, from + to −, or from − to +.

71. 1. Enclose the last three terms of $a-b+c-d+e$ in a parenthesis preceded by a − sign.

Result, $\qquad a-b-(-c+d-e)$.

In each of the following expressions, enclose the last three terms in a parenthesis preceded by a − sign:

2. $a+b+c+d$.

3. $3a-2b+5c-4d$.

4. m^3+5m^2-6m+3.

5. $x^3y-x^2y^2-xy^3+y^4$.

6. $x^4-3x^3+2x^2-5x-8$.

7. $a^2-b^2-c^2+2ab+2ac$.

8. In each of the above results, enclose the last two terms in an *inner* parenthesis preceded by a − sign.

V. MULTIPLICATION.

72. Multiplication, in Algebra, is the process of taking one quantity as many times as there are units in another.

Thus, the multiplication of a by b, which is expressed ab (Art. 10), signifies that the quantity a is to be taken b times.

73. The *Multiplicand* is the quantity to be multiplied or taken.

The *Multiplier* is the quantity which shows how many times it is to be taken.

The *Product* is the result of the operation.

The multiplicand and multiplier are called *factors.*

74. In Arithmetic, the product of two numbers is the same in whatever order they are taken; thus, we have 3×4 or 4×3, each equal to 12.

Similarly, in Algebra, we have $a \times b$ or $b \times a$, each equal to ab.

That is, *the product of the factors is the same in whatever order they are taken.*

75. Required the product of c and $a - b$.

In Arithmetic, if we wish to multiply 87 by 98, we may express the multiplier in the form $100 - 2$; we should then multiply 87 by 100, and afterwards by 2, and subtract the second result from the first.

Similarly, in Algebra, to multiply c by $a - b$, we should multiply c by a, and afterwards by b, and subtract the second result from the first. Thus, the required product is

$$ac - bc.$$

76. Required the product of $a - b$ and $c - d$.

As in Art. 75, we should first multiply $a - b$ by c, and

afterwards by d, and subtract the second result from the first.

The product of $a-b$ and c is $ac-bc$ (Art. 75).
The product of $a-b$ and d is $ad-bd$.

Subtracting the second result from the first, the required product is

$$ac-bc-ad+bd.$$

77. We observe, in the preceding article, that the product is formed by multiplying each term of the multiplicand by each term of the multiplier, with the following results in regard to signs:

The product of the terms $+a$ and $+c$ gives the term $+ac$.
The product of the terms $-b$ and $+c$ gives the term $-bc$.
The product of the terms $+a$ and $-d$ gives the term $-ad$.
The product of the terms $-b$ and $-d$ gives the term $+bd$.

From these considerations we may state what is called the **Rule of Signs** in Multiplication, as follows:

$+$ *multiplied by* $+$, *and* $-$ *multiplied by* $-$, *produce* $+$;
$+$ *multiplied by* $-$, *and* $-$ *multiplied by* $+$, *produce* $-$.

Or, as it is usually expressed with regard to the product of two terms,

Like signs produce $+$, *and unlike signs produce* $-$.

78. Required the product of $7a$ and $2b$.

Since the factors may be written in any order (Art. 74), we have

$$7a\times 2b=7\times 2\times a\times b=14ab.$$

That is, *the coefficient of the product is the product of the coefficients of the factors.*

79. Required the product of a^3 and a^2.

By Art. 13, $a^3=a\times a\times a$, and $a^2=a\times a$. Hence,

$$a^3\times a^2=a\times a\times a\times a\times a=a^5.$$

That is, *the exponent of a letter in the product is the sum of its exponents in the factors.*

Thus, $$a^5 \times a^3 \times a = a^{5+3+1} = a^9.$$

MULTIPLICATION OF MONOMIALS.

80. We derive from Arts. 77, 78, and 79 the following rule for the product of two monomials:

To the product of the coefficients annex the literal quantities, giving to each letter an exponent equal to the sum of its exponents in the factors. Make the product + when the factors have the same sign, and − when they have different signs.

EXAMPLES.

1. Multiply $2a^5$ by $7a^4$.

By the rule, $2a^5 \times 7a^4 = 14a^{5+4} = 14a^9$, *Ans.*

2. Multiply a^3b^2c by $-5a^2bd$.

$$a^3b^2c \times (-5a^2bd) = -5a^5b^3cd, \text{ Ans.}$$

3. Multiply $-7x^m$ by $5x^3$.

$$-7x^m \times 5x^3 = -35x^{m+3}, \text{ Ans.}$$

4. Multiply $-11x^m$ by $-8x^m$.

$$-11x^m \times (-8x^m) = 88x^{2m}, \text{ Ans.}$$

Multiply the following:

5. 13 by -19.

6. -18 by 12.

7. -22 by -51.

8. $15m^6n^6$ by $3mn$.

9. $17abc$ by $-8abc$.

10. $-17a^4c^2$ by $3a^2c^2$.

11. $-11n^2y$ by $-5n^6z$.

12. $-6a^2bc$ by a^3bm.

13. $-12a^2x$ by $-2x^2y$.

14. $-2a^mb^n$ by $5a^3b^n$.

15. $3a^3x^5y^2$ by $11ax^4y^5$.

16. $3a^mb^n$ by $-5a^nb^m$

It is evident from the Rule of Signs (Art. 77) that the product of three negative terms is negative; of four negative terms, positive; and so on.

Hence the product of three or more monomials will be positive or negative, according as the number of *negative* factors is odd or even.

17. Required the product of $-2a^2b^3$, $6bc^5$, and $-7c^2d$.

$$-2a^2b^3 \times 6bc^5 \times -7c^2d = 84a^2b^4c^7d, \textit{ Ans.}$$

In this case the product is positive, as there are two negative factors.

Multiply the following:

18. $5a$, $-6b$, and $7c$.

19. $-2a^2$, $-11a^3$, and $-9a$.

20. $-3ab^2$, $-2bc^2$, and $7cd^2$.

21. $4x^my^n$, $-x^ny^nz^5$, and $15y^2z^r$.

22. $-2a$, $-3a^2$, $-4a^3$, and $-5a^4$.

23. $-a^2bc$, $2b^2cd$, $-5a^3cd$, and $-3ab^5d^4$.

24. $-7m^nx^2$, $m^nx^ry^2$, $2x^3$, and $-8my^{2r}$.

25. $6xy^2$, $-x^3z$, $3y^4z^2$, $-2xz^5$, and $-4yz$.

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS.

81. In Art. 75 we showed that the product of $a-b$ and c was $ac-bc$. We have then the following rule for the product of a polynomial by a monomial:

Multiply each term of the multiplicand by the multiplier, and connect the results with their proper signs.

EXAMPLES.

1. Multiply $2x^2-5x-7$ by $8x^3$.

By the rule, the product is $16x^5-40x^4-56x^3$, *Ans.*

2. Multiply $-5ab^4$ by $3a^2b-4ab^3$.

$$\begin{array}{r} 3a^2b-4ab^3 \\ -\ \ 5ab^4 \\ \hline -15a^3b^5+20a^2b^7, \end{array}\ \textit{Ans.}$$

Multiply the following:

3. $3x-5$ by $4x$.

4. a^2b+ab^2 by $-ab$.

5. $8a^2bc-d$ by $5ad^2$.

6. x^2-2x-3 by $-4x$.

7. $-2x^3$ by $3x^2+6x-7$.

8. m^2+mn+n^2 by m^2n^2.

9. $-2m$ by $3m^2-5mn-n^2$.

10. $-x^4-10x^3+5$ by $-2x^3$.

11. $a^2+13ab-6b^2$ by $4ab^2$.

12. $-6a^2c$ by $5-6ac-8a^3$.

13. $5x^3-4x^2-3x+2$ by $-6x^5$.

14. a^2b^2 by $a^3-3a^2b+3ab^2-b^3$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS.

82. In Art. 76 it was shown that the product of $a-b$ and $c-d$ was $ac-bc-ad+bd$. We have then the following rule for the product of two polynomials:

Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.

EXAMPLES.

1. Multiply $3a-2b$ by $2a-5b$.

In accordance with the rule, we multiply $3a-2b$ by $2a$, and then by $-5b$, and add the partial products. A convenient arrangement of the work is shown below, similar terms being in the same vertical column.

$$\begin{array}{llll} 3a & -\ \ 2b & & \\ 2a & -\ \ 5b & & \\ \hline 6a^2 & -\ \ 4ab & & \\ & -15ab & +10b^2 & \\ \hline 6a^2 & -19ab & +10b^2, & \textit{Ans.} \end{array}$$

2. Multiply x^2+1-x^3-x by $x+1$.

It is convenient to have both multiplicand and multiplier arranged in the same order of powers (Art. 36), and to write the partial products in the same order.

Arranging the expressions according to the ascending powers of x, we have

$$\begin{array}{l} 1-x+x^2-x^3 \\ \underline{1+x} \\ 1-x+x^2-x^3 \\ \underline{\phantom{1-{}}x-x^2+x^3-x^4} \\ 1-x^4, \ \textit{Ans.} \end{array}$$

3. Multiply $6ab-8b^2+4a^2$ by $-4b^2+2a^2-3ab$.

Arranging according to the descending powers of a, we have

$$\begin{array}{lllll} 4a^2+6ab-8b^2 & & & & \\ \underline{2a^2-3ab-4b^2} & & & & \\ 8a^4+12a^3b & -16a^2b^2 & & & \\ -12a^3b & -18a^2b^2 & +24ab^3 & & \\ & \underline{-16a^2b^2} & \underline{-24ab^3} & \underline{+32b^4} & \\ 8a^4 & -50a^2b^2 & & +32b^4, & \textit{Ans.} \end{array}$$

Note. The correctness of the answers may be tested by working the examples with the multiplicand and multiplier interchanged.

Multiply the following:

4. $3x+2$ and $5x-7$.

5. $6x-5$ and $3-2x$.

6. $3a-2b$ and $-2a+4b$.

7. $3-5xy$ and $-6-10xy$

8. a^2+ab+b^2 and $b-a$.

9. $2a^2b-3ab^2$ and $5a^2b+6ab^2$.

10. $1+x+x^2+x^3$ and $ax-a$.

11. $3x^2-2xy-y^2$ and $2x-4y$.

12. $m^2-mn-3n^2$ and $2m^2-6mn$.

13. x^2+2x+1 and x^2-2x+3.

14. $5a^2 + 4b^2 - 3ab$ and $6a - 5b$.

15. $4x^3 + 6x - 7$ and $2x^2 - 3$.

16. $a + b - c$ and $a - b + c$.

17. $2x^2 - 3x + 5$ and $x^2 + x - 1$.

18. $3x^2 - 7x + 4$ and $2x^2 + 9x - 5$.

19. $2x^3 - 3x^2 - 5x - 1$ and $3x - 5$.

20. $6m - 2m^2 - 5 - m^3$ and $m^2 + 10 - 2m$.

21. $2x^3 + 5x^2 - 8x - 7$ and $4 - 5x - 3x^2$.

22. $a^3b - a^2b^2 - 4ab^3$ and $2a^2b - ab^2$.

23. $x^{m+2}y - 3xy^{n-1}$ and $4x^{m+5}y^2 - 4x^4y^n$.

24. $x^2 + y^2 - xy$ and $xy + y^2 + x^2$.

25. $2ab + b^2 + 4a^2$ and $4a^2 - 2ab + b^2$.

26. $6x^4 - 3x^3 - x^2 + 6x - 2$ and $2x^2 + x + 2$.

27. $m^4 - m^3n + m^2n^2 - mn^3 + n^4$ and $m^2 - 2mn - 3n^2$.

28. $27x^3 + 9x^2y + 3xy^2 + y^3$ and $9x^2 - 6xy + y^2$.

29. $a^3 - 3a^2b + 3ab^2 - b^3$ and $a^2 - 2ab + b^2$.

30. $x^2 + y^2 + z^2 - xy - yz - zx$ and $x + y + z$.

31. $2x^3 - 3x^2 + 5x - 1$ and $3x^3 - x^2 - 2x - 5$.

32. $ab + cd + ac + bd$ and $ab + cd - ac - bd$.

33. $2a^3 - 5a^2 - 6a + 4$ and $4a^3 + 10a^2 - 12a - 8$.

Find the product of the following:

34. $x - 3$, $x + 4$, and $x - 7$.

35. $a + b$, $a^2 - ab + b^2$, and $a^3 - b^3$.

36. $2m - 1$, $3m + 4$, and $6m - 5$.

37. $x + 1$, $3x - 2$, and $3x^2 - x - 2$.

38. $x^2 + x + 1$, $x^2 - x + 1$, and $x^4 - x^2 + 1$.

39. $a+b$, $a-b$, a^2+b^2, and a^4+b^4.

40. $m+1$, $m-1$, $m+2$, and $m-2$.

41. $2x-1$, $3x+2$, $4x-3$, and $5x+4$.

42. $a+b$, $a-b$, $a+2b$, and $a^3-2a^2b-ab^2+2b^3$.

83. The product of two or more polynomials may be ***indicated*** by enclosing each of them in a parenthesis, and writing them one after the other.

Thus, the product of $x+2$, $x-3$, and $2x-7$ is indicated by

$$(x+2)(x-3)(2x-7).$$

Similarly, the expression $(a+b+c)^2$ indicates that $a+b+c$ is to be multiplied by itself (Art. 13).

When the operations indicated are performed, the expression is said to be *expanded* or *simplified*.

EXAMPLES.

1. Simplify the expression $(a-2x)^2-2(x+3a)(a-x)$.

To simplify the expression, we should expand $(a-2x)^2$ and $2(x+3a)(a-x)$, and subtract the second result from the first.

$$(a-2x)^2 = a^2-4ax+4x^2$$
$$2(x+3a)(a-x) = 6a^2-4ax-2x^2$$

Subtracting the second result from the first, we have

$$a^2-4ax+4x^2-6a^2+4ax+2x^2 = 6x^2-5a^2, \ \textit{Ans.}$$

Simplify the following:

2. $(a+b+c+d)^2$.

3. $(a-b)(c-d)+(a-c)(b-d)$.

4. $(2x-3)^2+(1-x)(3x-9)$.

5. $(a+b+c)^2-(a-b+c)^2$.

6. $(2a-5b)^2-4(a-2b)(a-3b)$.

7. $(a-b)^2(a+b)^2$.

8. $(1+x)(1+x^4)(1-x+x^2-x^3)$.

9. $(1+a)^3-(1-a)(1+a^2)$.

10. $[x-(2y+3z)][x-(2y-3z)]$.

11. $(x+y)(x^3-y^3)[x^2-y(x-y)]$.

12. $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.

13. $(a+b+c)^2+(a-b-c)^2+(b-c-a)^2+(c-a-b)^2$.

14. $(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)$.

15. $x(x-2y)+y(y-2z)+z(z-2x)-(x-y-z)^2$.

16. $x(x+1)(x+2)(x+3)+1-(x^2+3x+1)^2$.

17. $(a+b+c)^2-(a-b-c)^2+(b-c-a)^2-(c-a-b)^2$.

18. $[(m+2n)^2-(2m-n)^2][(2m+n)^2-(m-2n)^2]$.

19. $(x+y+z)^3-(x^3+y^3+z^3)-3(y+z)(z+x)(x+y)$.

84. Since $(+a)(+b)=ab$, and $(-a)(-b)=ab$, it follows that in the indicated product of two factors *all the signs of both factors may be changed without altering the value of the expression.* Thus,

$$(a-b)(c-d) \text{ is equal to } (b-a)(d-c).$$

Similarly, we may show that in the indicated product of any number of factors, *the signs of any even number of factors may be changed without altering the value of the expression.*

Thus, $(a-b)(c-d)(e-f)$, by changing the signs of the second and third factors, may be written in the equivalent form $(a-b)(d-c)(f-e)$.

VI. DIVISION.

85. Division, in Algebra, is the process of finding one of two factors, when their product and the other factor are given.

Hence, Division is the converse of Multiplication.

Thus, the division of $14ab$ by $7a$, which is expressed $\frac{14ab}{7a}$ (Art. 15), signifies that we are to find a quantity which, when multiplied by $7a$, will produce $14ab$.

86. The *Dividend* is the product of the factors.
The *Divisor* is the given factor.
The *Quotient* is the required factor.

87. Since the dividend is the product of the divisor and quotient, it follows, from Art. 77, that:

If the divisor is $+$, and the quotient is $+$, the dividend is $+$.
If the divisor is $-$, and the quotient is $+$, the dividend is $-$.
If the divisor is $+$, and the quotient is $-$, the dividend is $-$.
If the divisor is $-$, and the quotient is $-$, the dividend is $+$.

In other words, if the dividend and divisor are both $+$, or both $-$, the quotient is $+$; and if the dividend and divisor are one $+$, and the other $-$, the quotient is $-$. Hence, in Division as in Multiplication,

Like signs produce $+$, and unlike signs produce $-$.

88. Required the quotient of $14ab$ divided by $7a$.

By Art 85, we are to find a quantity which, when multiplied by $7a$, will produce $14ab$. That quantity is evidently $2b$; hence

$$\frac{14ab}{7a} = 2b.$$

That is, *the coefficient of the quotient is the coefficient of the dividend, divided by the coefficient of the divisor.*

89. Required the quotient of a^5 divided by a^3.

We are to find a quantity which, when multiplied by a^3, will produce a^5. That quantity is evidently a^2; hence

$$\frac{a^5}{a^3} = a^2.$$

That is, *the exponent of a letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.*

For example, $$\frac{a^m}{a^n} = a^{m-n}.$$

DIVISION OF MONOMIALS.

90. We derive from Arts. 87, 88, and 89 the following rule for the division of monomials:

To the quotient of the coefficients annex the literal quantities, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor. Make the quotient + when the dividend and divisor have like signs, and − when they have unlike signs.

EXAMPLES.

1. Divide $54a^7$ by $-9a^4$.

By the rule, $$\frac{54a^7}{-9a^4} = -6a^{7-4} = -6a^3, \textit{ Ans.}$$

2. Divide $-2a^3b^2cd^4$ by abd^4.

$$\frac{-2a^3b^2cd^4}{abd^4} = -2a^2bc, \textit{ Ans.}$$

Note. A literal quantity having the same exponent in the dividend and divisor, as d^4 in Ex. 2, is *canceled* by the operation of division, and does not appear in the quotient.

3. Divide $-91x^my^nz^r$ by $-13x^ny^nz^3$.

$$\frac{-91x^my^nz^r}{-13x^ny^nz^3} = 7x^{m-n}z^{r-3}, \textit{ Ans.}$$

Divide the following:

4. 84 by -12.
5. -343 by 7.
6. -324 by -18.
7. 444 by -37.
8. $12a^5$ by $4a$.
9. $-a^2c$ by ac.
10. $2m^3n^4$ by $-mn^3$.
11. $-8x^2y^2$ by $-4x^2$.
12. $30a^5b^8$ by $5a^2b$.
13. $14m^3n^4$ by $-7mn^3$.
14. $-18x^2y^5z$ by $9x^2z$.
15. $-65a^3b^8c^3$ by $-5ab^2c^3$.
16. $72m^5n$ by $-12m^2$.
17. $12x^ay^b$ by $3x^cy^d$.
18. $-18a^mb$ by $6ab$.
19. $-144c^5d^7e^6$ by $-36c^2d^3e$.
20. $-3a^{m+2}$ by a^{m+1}.
21. $a^{m+n}b^{m+n}$ by a^mb^n.
22. $-91x^4y^3z^2$ by $-13x^3y^2$.
23. $18m^3n^4p^5$ by $-2m^2np^5$.

DIVISION OF POLYNOMIALS BY MONOMIALS.

91. The operation being simply the converse of Art. 81, we have the following rule:

Divide each term of the dividend by the divisor, and connect the results with their proper signs.

EXAMPLES.

1. Divide $9a^3b - 6a^4c + 12a^2bc$ by $-3a^2$.

By the rule,

$$\frac{9a^3b - 6a^4c + 12a^2bc}{-3a^2} = -3ab + 2a^2c - 4bc, \text{ } \textit{Ans.}$$

Divide the following:

2. $8a^3bc + 16a^5bc - 4a^7c^2$ by $4a^2c$.
3. $9x^4 + 27x^3 - 21x^2$ by $-3x^2$.
4. $30a^8 - 75a^4b$ by $15a^3$.
5. $2x^3y^2z - 12xy^2z^3$ by $-2xy^2z$.

6. $5a^2bc - 5ab^2c + 5abc^2$ by $-5abc$.

7. $4x^7 - 8x^6 - 14x^5 + 2x^4 - 6x^3$ by $2x^3$.

8. $-12a^pb^q - 30a^{12}b^5 + 108a^nb^n$ by $-6a^mb^n$.

9. $20x^4 - 12x^2 - 28x$ by $4x$.

10. $-a^2b^2c - ab^2c^2 + a^2bc^2$ by $-abc$.

11. $9a^5bc - 3a^2b + 18a^3bc$ by $-3ab$.

12. $15x^my^nz^r - 35x^{m+2}y^{2n}z$ by $5x^my^nz$.

13. $20a^4bc + 15abd^3 - 10a^2b$ by $-5ab$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS.

92. Required the quotient of $12 + 10x^3 - 11x - 21x^2$ divided by $2x^2 - 4 - 3x$.

Arranging both dividend and divisor according to the descending powers of x (Art. 37), we are to find a quantity which, when multiplied by the divisor, $2x^2 - 3x - 4$, will produce $10x^3 - 21x^2 - 11x + 12$.

It is evident, from Art. 82, that the term containing the highest power of x in the product, is the product of the terms containing the highest powers of x in the factors. Hence $10x^3$ is the product of $2x^2$ and the term containing the highest power of x in the quotient. Therefore the term containing the highest power of x in the quotient is $10x^3$ divided by $2x^2$, or $5x$.

Multiplying the divisor by $5x$, we have the product $10x^3 - 15x^2 - 20x$; which, when subtracted from the dividend, leaves the remainder $-6x^2 + 9x + 12$.

This remainder is the product of the divisor by the rest of the quotient; hence, to obtain the next term of the quotient, we proceed as before, regarding $-6x^2 + 9x + 12$ as a new dividend. Dividing the term containing the highest power of x, $-6x^2$, by the term containing the highest power

of x in the divisor, $2x^2$, we have -3 as the second term of the quotient.

Multiplying the divisor by -3, we have $-6x^2+9x+12$; which, when subtracted from the second dividend, leaves no remainder. Hence $5x-3$ is the required quotient.

It is customary to arrange the work as follows:

$$\begin{array}{r|ll}
10x^3-21x^2-11x+12 & 2x^2-3x-4, & \textit{Divisor.} \\
\underline{10x^3-15x^2-20x} & 5x\ -3, & \textit{Quotient.} \\
-\ \ 6x^2+\ \ 9x+12 & & \\
\underline{-\ \ 6x^2+\ \ 9x+12} & &
\end{array}$$

Note. We might have solved the example by arranging the dividend and divisor according to the *ascending* powers of x, in which case the quotient would have appeared in the form $-3+5x$.

93. From Art. 92, we derive the following rule for the division of polynomials:

Arrange both dividend and divisor in the same order of powers of some common letter.

Divide the first term of the dividend by the first term of the divisor, giving the first term of the quotient.

Multiply the whole divisor by this term, and subtract the product from the dividend, arranging the remainder in the same order of powers as the dividend and divisor.

Regard the remainder as a new dividend, and proceed as before; continuing until there is no remainder.

Note. The work may be verified by multiplying the quotient by the divisor, which should of course give the dividend.

EXAMPLES.

1. Divide $21x^2y^2-22xy-8$ by $3xy-4$.

$$\begin{array}{r|ll}
21x^2y^2-22xy-8 & 3xy-4 & \\
\underline{21x^2y^2-28xy} & 7xy+2, & \textit{Ans.} \\
6xy-8 & & \\
\underline{6xy-8} & &
\end{array}$$

2. Divide $8+18x^4-56x^2$ by $-6x^2+4+8x$.

Arranging according to the ascending powers of x,

$$\begin{array}{l|l}
8-56x^2+18x^4 & 4+8x-6x^2 \\
\underline{8+16x-12x^2} & 2-4x-3x^2, \textit{ Ans.} \\
\quad -16x-44x^2+18x^4 & \\
\quad \underline{-16x-32x^2+24x^3} & \\
\qquad\quad -12x^2-24x^3+18x^4 & \\
\qquad\quad \underline{-12x^2-24x^3+18x^4} &
\end{array}$$

3. Divide $9ab^2+a^3-9b^3-5a^2b$ by $3b^2+a^2-2ab$.

Arranging according to the descending powers of a,

$$\begin{array}{r}
a^2-2ab+3b^2)\,a^3-5a^2b+9ab^2-9b^3\,(a-3b, \textit{ Ans.} \\
\underline{a^3-2a^2b+3ab^2} \qquad\qquad\qquad\qquad \\
-3a^2b+6ab^2-9b^3 \qquad\qquad\qquad \\
\underline{-3a^2b+6ab^2-9b^3} \qquad\qquad\qquad
\end{array}$$

Divide the following:

4. $6x^2-x-35$ by $3x+7$.

5. $2-3ax-2a^2x^2$ by $1-2ax$.

6. $a^2-4ab+4b^2$ by $a-2b$.

7. $59x-56-15x^2$ by $3x-7$.

8. $3b^3+3ab^2-4a^2b-4a^3$ by $b+a$.

9. $2a^3x-2ax^3$ by $ax-a^2$.

10. $18x^3-5x+1$ by $6x^2+2x-1$.

11. $8m^3+35-36m$ by $5+2m$.

12. $27x^3+y^3$ by $3x+y$.

13. $16m^4-1$ by $2m-1$.

14. $a^2-b^2+c^2-2ac$ by $a+b-c$.

15. $8a^3+36a^2b+54ab^2+27b^3$ by $2a+3b$.

16. $x^4+y^4+x^2y^2$ by x^2+y^2+xy.

17. $2x^4 - 19x^3 + 9$ by $2x^3 + 6x^2 - x - 3$.

18. $8m^3 + 3n^3 - 4m^2n - 6mn^2$ by $2m - n$.

19. $4x^4 - 8x^3 - 6x^2 + 24$ by $2x - 4$.

20. $23x^2 - 48 + 6x^4 - 2x - 31x^3$ by $6 + 3x^2 - 5x$.

21. $4a^5 + 27 - a^3$ by $9 - 3a^3 + 4a^2 + 2a^4 - 6a$.

22. $x^4 - 9x^2 - 6xy - y^2$ by $x^2 + 3x + y$.

23. $a^8 - 81b^4$ by $a^2 + 3b$.

24. $x^2 - y^2 + 2yz - z^2$ by $x + y - z$.

25. $3x^4 - 14x^2 + 8$ by $x - 2$.

26. $y^6 + x^5y$ by $x + y$.

27. $15m^4 + 50m^2 + 15 - 32m - 32m^3$ by $3m^2 + 5 - 4m$.

28. $1 + 4x^3 + 3x^4$ by $(x + 1)^2$.

29. $21a^5 - 21b^5$ by $7a - 7b$.

30. $64x^4 + 1$ by $8x^2 - 4x + 1$.

31. $50x + 9x^4 + 24 - 67x^2$ by $x + x^2 - 6$.

32. $x^4 + y^4 - 4xy^3 - 4x^3y + 6x^2y^2$ by $x^2 + y^2 - 2xy$.

33. $x^4 - 4x^3 + 2x^2 + 4x + 1$ by $(x - 1)^2 - 2$.

34. $9x^4 + 4y^4 - 37x^2y^2$ by $3x^2 - 2y^2 + 5xy$.

35. $a^4 + a^2b^2 + 25b^4$ by $(a - b)(a - 5b) + 3ab$.

36. $3x^2 + 4x + 6x^5 - 11x^3 - 4$ by $3x^2 - 4$.

37. $6x^5 + 15x^3 + 51x - 18$ by $2x^3 - 4x^2 + 7x - 2$.

38. $2x^4 - 11x - 4x^2 - 12 - 3x^3$ by $4 + 2x^2 + x$.

39. $m^5 - 48 - 17m^3 + 52m + 12m^2$ by $m - 2 + m^2$.

40. $x^{n+1} + x^ny - xy^n - y^{n+1}$ by $x^n - y^n$.

41. $x^5y - xy^5$ by $x^3 + y^3 + xy^2 + x^2y$.

42. $x^5 - 6x^2 - x - 6$ by $x^2 + 2x + 3$.

43. $2a^5 + 53a^2b^3 - 49b^5 - 7a^3b^2 - 9a^4b$ by $2a^2 - 5ab - 7b^2$.

44. $x^6 - 6x^4 + 5x^2 - 1$ by $x^3 + 2x^2 - x - 1$.

45. $2x^2 - 6y^2 - 12z^2 + xy - 2xz + 17yz$ by $2x + 4z - 3y$.

46. $a^{2n} - b^{2m} + 2b^mc^r - c^{2r}$ by $a^n + b^m - c^r$.

47. $x^6 - 1 - 6x^4 - 3x^2$ by $-2x^2 - x + x^3 - 1$.

48. $12a^5 - 14a^4b + 10a^3b^2 - a^2b^3 - 8ab^4 + 4b^5$
by $6a^3 - 4a^2b - 3ab^2 + 2b^3$.

The operation of division may be abridged in certain cases by the use of parentheses.

49. Divide $(a^2 + ab)x^2 + (2ac + bc + ad)x + c(c + d)$
by $ax + c$.

$$
\begin{array}{l|l}
(a^2+ab)x^2 + (2ac + bc + ad)x + c(c+d) & ax + c \\
\cline{2-2}
\underline{(a^2+ab)x^2 + (\ \ ac + bc \qquad\ )x \qquad\qquad\quad} & (a+b)x + (c+d), \\
\qquad\qquad\quad (\ \ ac \qquad + ad)x + c(c+d) & \qquad\qquad\qquad \textit{Ans.} \\
\qquad\qquad\quad \underline{(\ \ ac \qquad + ad)x + c(c+d)} &
\end{array}
$$

Divide the following:

50. $x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$
by $x^2 + (b+c)x + bc$.

51. $(b + c)a^2 + (b^2 + 3bc + c^2)a + bc(b + c)$ by $a + b + c$.

52. $(x + y)^2 - 5(x + y) + 6$ by $(x + y) - 2$.

53. $(a + b)^3 + 1$ by $(a + b) + 1$.

54. $x^3 + (a + b - c)x^2 + (ab - bc - ca)x - abc$
by $x^2 + (b - c)x - bc$.

55. $(m - n)^4 - 2(m - n)^2 + 1$
by $(m - n)^2 - 2(m - n) + 1$.

56. $x^3 + (a - b + c)x^2 + (ac - ab - bc)x - abc$ by $x + c$.

57. $x^4 + (3 - b)x^3 + (c - 3b - 2)x^2 + (2b + 3c)x - 2c$
by $x^2 + 3x - 2$.

58. $a^2(b + c) + a(b^2 + bc + c^2) - bc(b + c)$ by $a + b + c$.

VII. FORMULÆ.

94. A **Formula** is the algebraic expression of a general rule.

95. The following results are of great importance in abridging algebraic operations:

$$\begin{array}{l}
a+b\\
\underline{a+b}\\
a^2+ab\\
\underline{\quad\;\; ab+b^2}\\
a^2+2ab+b^2
\end{array}
\qquad
\begin{array}{l}
a-b\\
\underline{a-b}\\
a^2-ab\\
\underline{\quad -ab+b^2}\\
a^2-2ab+b^2
\end{array}
\qquad
\begin{array}{l}
a+b\\
\underline{a-b}\\
a^2+ab\\
\underline{\quad -ab-b^2}\\
a^2\qquad\;\; -b^2
\end{array}$$

In the first case, we have $(a+b)^2=a^2+2ab+b^2$. (1)

That is, *the square of the sum of two quantities is equal to the square of the first, plus twice the product of the two, plus the square of the second.*

In the second case, we have $(a-b)^2=a^2-2ab+b^2$. (2)

That is, *the square of the difference of two quantities is equal to the square of the first, minus twice the product of the two, plus the square of the second.*

In the third case, we have $(a+b)(a-b)=a^2-b^2$. (3)

That is, *the product of the sum and difference of two quantities is equal to the difference of their squares.*

EXAMPLES.

96. 1. Square $3a+2bc$.

The square of the first term is $9a^2$, twice the product of the terms is $12abc$, and the square of the second term is $4b^2c^2$. Hence, by formula (1),

$$(3a+2bc)^2=9a^2+12abc+4b^2c^2.$$

Note. The following rule for the square of a monomial is evident from the above:

Square the coefficient, and multiply the exponent of each letter by 2.

Thus, the square of $5a^2b$ is $25a^4b^2$.

2. Square $4x - 5$.

By formula (2), $(4x-5)^2 = 16x^2 - 40x + 25$, *Ans.*

3. Multiply $6a^2 + b$ by $6a^2 - b$.

By formula (3), $(6a^2+b)(6a^2-b) = 36a^4 - b^2$, *Ans.*

Write by inspection the values of the following:

4. $(x-4)^2$.
5. $(3+a)^2$.
6. $(x+3)(x-3)$.
7. $(3a+5)^2$.
8. $(2x+1)(2x-1)$.
9. $(7-2x)^2$.
10. $(2m+3n)^2$.
11. $(4ab-x)^2$.
12. $(5+7x)(5-7x)$.
13. $(x^4-y^2)^2$.
14. $(3x+11)(3x-11)$.
15. $(x^2y+4)^2$.
16. $(3x^3+13)^2$.
17. $(6a^2-b^2c)^2$.
18. $(5a+7b^2)(5a-7b^2)$.
19. $(13ab+5ac)^2$.
20. $(x^3+5x)(x^3-5x)$.
21. $(1-12xyz)^2$.
22. $(4x^2+3y^5)(4x^2-3y^5)$.
23. $(10x^3+9x^2)^2$.
24. $(4a^p-5b^q)^2$.
25. $(a^m+a^n)(a^m-a^n)$.
26. $(7x^3+11x)^2$.
27. $(5a^m-a^n)^2$.

28. Multiply $a+b+c$ by $a+b-c$.

$$\begin{aligned}(a+b+c)(a+b-c) &= [(a+b)+c][(a+b)-c] \\ &= (a+b)^2 - c^2, \quad \text{by formula (3)} \\ &= a^2 + 2ab + b^2 - c^2, \textit{ Ans.}\end{aligned}$$

29. Multiply $a+b-c$ by $a-b+c$.

$$\begin{aligned}(a+b-c)(a-b+c)&=[a+(b-c)][a-(b-c)]\\&=a^2-(b-c)^2\\&=a^2-(b^2-2bc+c^2)\\&=a^2-b^2+2bc-c^2,\ \textit{Ans.}\end{aligned}$$

Expand the following:

30. $(x+y+z)(x-y+z)$.

31. $(x+y+z)(x-y-z)$.

32. $(1+a-b)(1-a+b)$.

33. $(x^2+x+1)(x^2-x-1)$.

34. $(a+b-c)(a-b-c)$.

35. $(a^2+2a+1)(a^2-2a+1)$.

36. $(x^2+2x-3)(x^2-2x+3)$.

37. $(m^2+mn+n^2)(m^2-mn+n^2)$.

97. We find by multiplication:

$$\begin{array}{r}x+5\\ \underline{x+3}\\ x^2+5x\\ \underline{+3x+15}\\ x^2+8x+15\end{array}\qquad\qquad\begin{array}{r}x-5\\ \underline{x-3}\\ x^2-5x\\ \underline{-3x+15}\\ x^2-8x+15\end{array}$$

$$\begin{array}{r}x+5\\ \underline{x-3}\\ x^2+5x\\ \underline{-3x-15}\\ x^2+2x-15\end{array}\qquad\qquad\begin{array}{r}x-5\\ \underline{x+3}\\ x^2-5x\\ \underline{+3x-15}\\ x^2-2x-15\end{array}$$

We observe in these products the following laws:

I. The coefficient of x is the algebraic sum of the numbers in the factors.

II. The last term is the product of the numbers.

By aid of the above laws the product of two binomials of the form $x+a$, $x+b$ may be written by inspection.

1. Required the value of $(x-8)(x+5)$.

The coefficient of x is -3; and the last term is -40.

Hence, $$(x-8)(x+5)=x^2-3x-40, \textit{ Ans.}$$

EXAMPLES.

Write by inspection the values of the following:

2. $(x+7)(x+5)$.

3. $(x-3)(x-4)$.

4. $(x+8)(x-2)$.

5. $(x-3)(x+1)$.

6. $(x-5)(x+6)$.

7. $(x+1)(x+12)$.

8. $(x-7)(x+2)$.

9. $(x-8)(x-6)$.

10. $(x+9)(x-5)$.

11. $(x-8)(x-9)$.

12. $(x+4m)(x+6m)$.

13. $(x-5a)(x+a)$.

14. $(a+b)(a-4b)$.

15. $(a+5b)(a+8b)$.

16. $(x^2-3)(x^2-7)$.

17. $(x^3+2a)(x^3-6a)$.

98. The following results may be verified by division:

(1) $\dfrac{a^2-b^2}{a+b}=a-b.$

(2) $\dfrac{a^2-b^2}{a-b}=a+b.$

(3) $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2.$

(4) $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2.$

Formulæ (3) and (4) may be stated in words as follows:

If the sum of the cubes of two quantities be divided by the sum of the quantities, the quotient is equal to the square of the first quantity, minus the product of the two, plus the square of the second.

If the difference of the cubes of two quantities be divided by the difference of the quantities, the quotient is equal to the square of the first quantity, plus the product of the two, plus the square of the second.

EXAMPLES.

1. Divide $36y^2z^4 - 9$ by $6yz^2 + 3$.

By formula (1), $\dfrac{36y^2z^4 - 9}{6yz^2 + 3} = 6yz^2 - 3$, *Ans.*

2. Divide $1 + 8a^3$ by $1 + 2a$.

By formula (3), $\dfrac{1 + 8a^3}{1 + 2a} = 1 - 2a + 4a^2$, *Ans.*

3. Divide $27a^3 - b^3$ by $3a - b$.

By formula (4), $\dfrac{27a^3 - b^3}{3a - b} = 9a^2 + 3ab + b^2$, *Ans.*

EXAMPLES.

Write by inspection the values of the following:

4. $\dfrac{x^2 - 81}{x - 9}$.

5. $\dfrac{25 - 16a^2}{5 + 4a}$.

6. $\dfrac{x^3 + 1}{x + 1}$.

7. $\dfrac{1 - m^3}{1 - m}$.

8. $\dfrac{a^3 - 8}{a - 2}$.

9. $\dfrac{27 + x^3}{3 + x}$.

10. $\dfrac{x^6 - 16x^2}{x^3 + 4x}$.

11. $\dfrac{x^3 - 64}{x - 4}$.

12. $\dfrac{1 - 8m^3}{1 - 2m}$.

13. $\dfrac{a^3 + 343}{a + 7}$.

14. $\dfrac{x^6 - y^6}{x^2 - y^2}$.

15. $\dfrac{27 + x^3y^6}{3 + xy^2}$.

16. $\dfrac{49a^2 - 121b^4}{7a - 11b^2}$.

17. $\dfrac{64m^3 + n^6}{4m + n^2}$.

18. $\dfrac{x^3 + 125y^3}{x + 5y}$.

Divide the following:

19. $27x^3y^3 - 64z^3$ by $3xy - 4z$.

20. $25a^4 - 81b^2c^6$ by $5a^2 - 9bc^3$.

21. $343 + 125x^3y^3$ by $7 + 5xy$.

22. $64m^3 - 216n^6$ by $4m - 6n^2$.

23. $729x^9y^3 + 512z^6$ by $9x^3y + 8z^2$.

99. By actual division we obtain:

$$\frac{a^4 - b^4}{a + b} = a^3 - a^2b + ab^2 - b^3.$$

$$\frac{a^4 - b^4}{a - b} = a^3 + a^2b + ab^2 + b^3.$$

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

$$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4; \text{ etc.}$$

In these results we observe the following laws:

I. The number of terms is the same as the exponent of a in the dividend.

II. The exponent of a in the first term is less by 1 than the exponent of a in the dividend, and decreases by 1 in each succeeding term.

III. The exponent of b in the second term is 1, and increases by 1 in each succeeding term.

IV. The terms are all positive when the divisor is $a - b$, and are alternately positive and negative when the divisor is $a + b$.

100. In connection with Art. 99, the following principles are of great importance:

If n is any whole number,

(1) *$a^n + b^n$ is divisible by $a + b$ if n is odd, and by neither $a + b$ nor $a - b$ if n is even.*

(2) *$a^n - b^n$ is divisible by $a - b$ if n is odd, and by both $a + b$ and $a - b$ if n is even.*

EXAMPLES.

101. 1. Divide $a^7 - b^7$ by $a - b$.

Applying the laws of Art. 99, we have

$$\frac{a^7 - b^7}{a - b} = a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6, \textit{ Ans.}$$

2. Divide $x^4 - 81$ by $x + 3$.

Since $81 = 3^4$, we have

$$\frac{x^4 - 3^4}{x + 3} = x^3 - 3x^2 + 3^2x - 3^3 = x^3 - 3x^2 + 9x - 27, \textit{ Ans.}$$

Write by inspection the values of the following:

3. $\dfrac{a^6 - b^6}{a - b}$.

4. $\dfrac{x^6 - y^6}{x + y}$.

5. $\dfrac{m^7 + n^7}{m + n}$.

6. $\dfrac{m^7 - n^7}{m - n}$.

7. $\dfrac{1 - x^4}{1 - x}$.

8. $\dfrac{x^4 - 16}{x - 2}$.

9. $\dfrac{1 - a^5}{1 - a}$.

10. $\dfrac{a^5 + 1}{a + 1}$.

11. $\dfrac{1 - n^6}{1 - n}$.

12. $\dfrac{x^4 - 81}{x - 3}$.

13. $\dfrac{x^5 - 32}{x - 2}$.

14. $\dfrac{a^6 - 64}{a + 2}$.

15. $\dfrac{a^{10} + b^{10}}{a^2 + b^2}$.

16. $\dfrac{x^7 - 128}{x - 2}$.

17. $\dfrac{x^5 + 243}{x + 3}$.

Divide the following:

18. $m^4 - 16n^8$ by $m - 2n^2$.

19. $x^8 - y^8z^8$ by $x - yz$.

20. $32a^5 + b^5$ by $2a + b$.

21. $m^5 - 243n^5$ by $m - 3n$.

22. $256x^4 - y^8$ by $4x + y^2$.

VIII. FACTORING.

102. **Factoring** is the process of resolving a quantity into its factors. (Art. 11.)

103. The factoring of monomials may be performed by inspection; thus,

$$12a^3b^2c = 2\cdot 2\cdot 3\,aaabbc.$$

A polynomial is not always factorable; but there are certain forms which can always be factored, the more important of which will be considered in the succeeding articles.

CASE I.

104. *When the terms of the polynomial have a common monomial factor.*

1. Factor a^3+3a.

Each term contains the monomial factor a.

Dividing the expression by a, we have a^2+3. Hence,

$$a^3+3a = a(a^2+3), \text{ *Ans.*}$$

2. Factor $14xy^4-35x^3y^2$.

$$14xy^4-35x^3y^2 = 7xy^2(2y^2-5x^2), \text{ *Ans.*}$$

EXAMPLES.

Factor the following:

3. x^2+5x.
4. $3m^3-12m^2$.
5. $16a^4-12a$.
6. $27c^4d^2+9c^3d$.
7. $60m^2n^4-12m^3$.
8. $5x^3+10x^2+15x$.
9. $a^5-2a^4+3a^3-a^2$.
10. $36x^3y-60x^2y^4-84x^4y^2$.
11. $21m^3n+35mn^3-14mn$.
12. $84x^2y^3-140x^3y^4+70x^4y^5$.

13. Factor the sum of $54a^4b^3$, $-72a^3c^2$, and $-90a^2d$.

14. Factor the sum of $96c^4d^5$, $120c^2d^7$, and $-144c^5d^4$.

Case II.

105. *When the polynomial consists of four terms, of which the first two and the last two have a common binomial factor.*

1. Factor $am - bm + an - bn$.

Factoring the first two and last two terms as in Case I, we have

$$m(a-b)+n(a-b).$$

Each term now contains the binomial factor $a-b$. Dividing the expression by $a-b$, we obtain $m+n$. Hence,

$$am - bm + an - bn = (a-b)(m+n), \; Ans.$$

2. Factor $am - bm - an + bn$.

$$\begin{aligned} am - bm - an + bn &= am - bm - (an - bn) \\ &= m(a-b) - n(a-b) \\ &= (a-b)(m-n), \; Ans. \end{aligned}$$

Note. If the third term is negative, as in Ex. 2, it is convenient, before factoring, to enclose the last two terms in a parenthesis preceded by a $-$ sign.

EXAMPLES.

Factor the following:

3. $ab + bx + ay + xy$.
4. $ac - cm + ad - dm$.
5. $x^2 + 2x - xy - 2y$.
6. $x^2 - ax - bx + ab$.
7. $a^3 - a^2b + ab^2 - b^3$.
8. $a^3 - a^2b - ab^2 + b^3$.
9. $x^2 + ax - bx - ab$.
10. $mx^2 - my^2 + nx^2 - ny^2$.
11. $x^3 + x^2 + x + 1$.
12. $6x^3 + 4x^2 - 9x - 6$.
13. $8cx - 12cy + 2dx - 3dy$.
14. $6n - 21m^2n - 8m + 28m^3$.

106. If a quantity can be resolved into two equal factors, it is said to be a *perfect square*, and one of the equal factors is called its square root.

Thus, since $9a^4b^2$ equals $3a^2b \times 3a^2b$, it is a perfect square, and $3a^2b$ is its square root.

Note. $9a^4b^2$ also equals $-3a^2b \times -3a^2b$, so that its square root is either $3a^2b$ or $-3a^2b$. In the examples in this chapter we shall consider the *positive* square root only.

107. The following rule for extracting the square root of a monomial is evident from Art. 106:

Extract the square root of the coefficient, and divide the exponent of each letter by 2.

For example, the square root of $25x^6y^8z^2$ is $5x^3y^4z$.

108. It follows from Art. 95 that a trinomial is a perfect square when its first and last terms are perfect squares and positive, and the second term is twice the product of their square roots.

Thus, $4x^2 - 12xy + 9y^2$ is a perfect square.

109. To find the square root of a perfect trinomial square, we take the converse of the rules of Art. 95:

Extract the square roots of the first and last terms, and connect the results by the sign of the second term.

Thus, let it be required to find the square root of

$$4x^2 - 12xy + 9y^2.$$

The square root of the first term is $2x$, and of the last term $3y$; and the sign of the second term is —. Hence the required square root is

$$2x - 3y.$$

Case III.

110. *When a trinomial is a perfect square* (Art. 108).

1. Factor $a^2 + 2ab^2 + b^4$.

By Art. 109, the square root of the expression is $a + b^2$. Hence,

$$a^2 + 2ab^2 + b^4 = (a + b^2)(a + b^2), \text{ or } (a + b^2)^2, \textit{ Ans.}$$

2. Factor $4x^2 - 12xy + 9y^2$.

$$4x^2 - 12xy + 9y^2 = (2x - 3y)(2x - 3y)$$
$$= (2x - 3y)^2, \textit{ Ans.}$$

Note. The given expression may be written $9y^2 - 12xy + 4x^2$; whence,

$$9y^2 - 12xy + 4x^2 = (3y - 2x)(3y - 2x) = (3y - 2x)^2;$$

which is another form of the answer.

EXAMPLES.

Factor the following:

3. $x^2 + 2xy + y^2$.
4. $4 + 4m + m^2$.
5. $x^2 - 14x + 49$.
6. $a^2 - 10a + 25$.
7. $y^2 + 2y + 1$.
8. $m^2 - 2m + 1$.
9. $x^4 + 12x^2 + 36$.
10. $n^6 - 20n^3 + 100$.
11. $x^2y^2 + 16xy + 64$.
12. $1 - 10ab + 25a^2b^2$.
13. $16m^2 - 8am + a^2$.
14. $a^4 + 2a^3 + a^2$.
15. $x^6 - 4x^4 + 4x^2$.
16. $36m^2 - 36mn + 9n^2$.
17. $4a^2 + 44ab + 121b^2$.
18. $x^6 + 8x^5 + 16x^4$.
19. $a^2b^4 + 18ab^2c + 81c^2$.
20. $25x^2 - 70xyz + 49y^2z^2$.
21. $9x^8 - 66x^6 + 121x^4$.
22. $9a^4 + 60a^2bc^2d + 100b^2c^4d^2$.
23. $64x^8 - 160x^7 + 100x^6$.
24. $4a^4b^2 + 52a^3b^3 + 169a^2b^4$.
25. $16x^4 - 120mnx^2 + 225m^2n^2$.
26. $(a - b)^2 + 2(a - b) + 1$.
27. $(x + y)^2 - 16(x + y) + 64$.
28. $(x^2 - x)^2 + 6(x^2 - x) + 9$.

Case IV.

111. *When an expression is the difference of two perfect squares.*

Comparing with the third case of Art. 95, we see that such an expression is the product of the sum and difference of two quantities.

Therefore, to obtain the factors, we take the converse of the rule of Art. 95:

Extract the square root of the first term and of the last term; add the results for one factor, and subtract the second result from the first for the other.

1. Factor $36x^2 - 49y^2$.

The square root of the first term is $6x$, and of the last term $7y$. Hence, by the rule,

$$36x^2 - 49y^2 = (6x + 7y)(6x - 7y),\ \textit{Ans.}$$

2. Factor $(2x - 3y)^2 - (x - y)^2$.

$$\begin{aligned}(2x - 3y)^2 - (x - y)^2 \\ &= [(2x - 3y) + (x - y)][(2x - 3y) - (x - y)] \\ &= (2x - 3y + x - y)(2x - 3y - x + y) \\ &= (3x - 4y)(x - 2y),\ \textit{Ans.}\end{aligned}$$

EXAMPLES.

Factor the following:

3. $x^2 - y^2$.	7. $9x^2 - 16y^2$.	11. $49m^2 - 100n^6$.
4. $x^2 - 1$.	8. $25a^2 - b^4$.	12. $36x^4 - 81y^2$.
5. $4 - a^2$.	9. $1 - 49x^2y^2$.	13. $64a^2 - 121b^2c^2$.
6. $9m^2 - 4$.	10. $a^2b^6 - c^4d^8$.	14. $144x^2y^4 - 225z^6$.

15. $(a+b)^2-(c+d)^2$.

16. $(a-c)^2-b^2$.

17. $m^2-(x-y)^2$.

18. $m^4-(m-1)^2$.

19. $(x-c)^2-(y-d)^2$.

20. $(a-3)^2-(b+2)^2$.

21. $(2x+m)^2-(x-m)^2$.

22. $(3a+5)^2-(2a-3)^2$.

It is sometimes possible to express a polynomial in the form of the difference of two perfect squares, when it may be factored by the rule of Case IV.

23. Factor $2mn+m^2-1+n^2$.

The expression may be written $m^2+2mn+n^2-1$, which, by Case III., is equivalent to $(m+n)^2-1$. Hence, by the rule,

$$(m+n)^2-1=(m+n+1)(m+n-1), \textit{ Ans.}$$

24. Factor $2xy+1-x^2-y^2$.

$$2xy+1-x^2-y^2=1-x^2+2xy-y^2=1-(x^2-2xy+y^2).$$

By Case III., this may be written $1-(x-y)^2$. Hence the factors are

$$[1+(x-y)][1-(x-y)]=(1+x-y)(1-x+y), \textit{ Ans.}$$

25. Factor $2xy+b^2-x^2-2ab-y^2+a^2$.

$$\begin{aligned}2xy+b^2-x^2-2ab-y^2+a^2 \\ &=a^2-2ab+b^2-x^2+2xy-y^2\\ &=a^2-2ab+b^2-(x^2-2xy+y^2)\\ &=(a-b)^2-(x-y)^2, \text{ by Case III.}\\ &=[(a-b)+(x-y)][(a-b)-(x-y)]\\ &=(a-b+x-y)(a-b-x+y), \textit{ Ans.}\end{aligned}$$

Factor the following:

26. $x^2+2xy+y^2-4$.

27. $a^2-2ab+b^2-c^2$.

28. $a^2-b^2+2bc-c^2$.

29. $a^2-b^2-2bc-c^2$.

30. $c^2 - 1 + d^2 + 2cd.$

31. $9 - x^2 - y^2 + 2xy.$

32. $4b - 1 - 4b^2 + 4m^4.$

33. $4a^2 + b^2 - 9d^2 - 4ab.$

34. $a^2 - 2am + m^2 - b^2 - 2bn - n^2.$

35. $x^2 - y^2 + c^2 - d^2 - 2cx + 2dy.$

36. $a^2 - b^2 + m^2 - n^2 + 2am + 2bn.$

37. $a^2 - b^2 + c^2 - d^2 + 2ac - 2bd.$

Case V.

112. *When an expression is a trinomial of the form $x^2 + ax + b$.*

In Art. 97 we derived a rule for the product of two binomials of the form $x + a$, $x + b$, by considering the following cases in multiplication:

1. $(x + 5)(x + 3) = x^2 + 8x + 15.$
2. $(x - 5)(x - 3) = x^2 - 8x + 15.$
3. $(x + 5)(x - 3) = x^2 + 2x - 15.$
4. $(x - 5)(x + 3) = x^2 - 2x - 15.$

In certain cases it is possible to reverse the operation, and resolve a trinomial of the form $x^2 + ax + b$ into the product of two binomial factors.

The first term of each factor will obviously be x; and to obtain the second terms, we take the converse of the rule of Art. 97:

Find two numbers whose algebraic sum is the coefficient of x, and whose product is the last term.

Thus, let it be required to factor $x^2 - 5x - 24$.

The coefficient of x is -5, and the last term is -24; we are then to find two numbers whose algebraic sum is -5, and product -24. By inspection we determine that the numbers are -8 and 3. Hence,

$$x^2 - 5x - 24 = (x - 8)(x + 3).$$

113. The work of finding the numbers may be abridged by the following considerations:

1. When the last term of the product is $+$, as in Exs. 1 and 2, the coefficient of x is the *sum* of the numbers; both numbers being $+$ when the second term is $+$, and $-$ when the second term is $-$.

2. When the last term of the product is $-$, as in Exs. 3 and 4, the coefficient of x is the *difference* of the numbers (disregarding signs); the greater number having the same sign as the second term, and the smaller number the opposite sign.

We may embody these observations in two rules, which will be found more convenient than the rule of Art. 112 in the solution of examples:

I. *If the last term is $+$, find two numbers whose sum is the coefficient of x, and whose product is the last term; and give to both numbers the sign of the second term.*

II. *If the last term is $-$, find two numbers whose difference is the coefficient of x, and whose product is the last term; give to the greater number the sign of the second term, and to the smaller number the opposite sign.*

Note. By the expressions "coefficient of x" and "last term," in the above rules, we understand their *absolute values*, without regard to sign.

EXAMPLES.

114. 1. Factor $x^2+14x+45$.

According to Rule I., we find two numbers whose sum is 14, and product 45. The numbers are 9 and 5; and as the second term is $+$, both numbers are $+$. Hence,

$$x^2+14x+45=(x+9)(x+5),\ \textit{Ans.}$$

2. Factor x^2-6x+5.

By Rule I., we find two numbers whose sum is 6, and

product 5. The numbers are 5 and 1; and as the second term is $-$, both numbers are $-$. Hence,

$$x^2-6x+5=(x-5)(x-1), \textit{ Ans.}$$

3. Factor $x^2+5x-14$.

By Rule II., we find two numbers whose difference is 5, and product 14. The numbers are 7 and 2; and as the second term is $+$, the greater number is $+$, and the smaller number $-$. Hence,

$$x^2+5x-14=(x+7)(x-2), \textit{ Ans.}$$

4. Factor $x^2-5x-24$.

By Rule II., we find two numbers whose difference is 5, and product 24. The numbers are 8 and 3; and as the second term is $-$, the greater number is $-$, and the smaller number $+$. Hence,

$$x^2-5x-24=(x-8)(x+3), \textit{ Ans.}$$

Factor the following:

5. x^2+5x+6.

6. x^2-3x+2.

7. y^2+2y-8.

8. $m^2-7m-30$.

9. $a^2-11a+18$.

10. x^2+x-6.

11. c^2+9c+8.

12. $y^2-2y-35$.

13. $a^2+13a-48$.

14. $x^2-10x+21$.

15. $x^2+13x+36$.

16. n^2-n-90.

17. $x^2-6x-16$.

18. $m^2+16m+63$.

19. $a^2-15a+44$.

20. $y^2+7y-60$.

21. $x^2-11x+10$.

22. $m^2+2m-80$.

23. $n^2+23n+102$.

24. $x^2-9x-90$.

25. $a^2-11a-26$.

26. x^2+x-42.

27. $c^2-18c+32$.

28. $m^2-8m-33$.

29. $x^2+20x+75$.

30. $x^2+4x-96$.

31. $y^2-17y-110$.

32. $x^2-19x+78$.

33. $x^2+7x-98$.

34. $a^2+22a+105$.

35. $x^2-23x+130$.

36. a^4+10a^2-144.

37. x^4-19x^2-120.

38. c^6+12c^3+11.

39. $x^2y^6+2xy^3-120$.

40. $a^2b^4-7ab^2-144$.

41. $n^2x^2+25nx+100$.

42. y^8-20y^4+91.

43. $a^4b^4-2a^2b^2-48$.

44. m^4+26m^2-87.

45. Factor $x^4+5abx^2-84a^2b^2$.

We find two numbers whose difference is 5, and product 84. The numbers are 12 and 7; and, by the rule, the greater is +, and the smaller –. Hence,

$$x^4+5abx^2-84a^2b^2=(x^2+12ab)(x^2-7ab),\ \textit{Ans.}$$

46. Factor $1-6a-27a^2$.

The numbers whose difference is 6, and product 27, are 9 and 3. Hence,

$$1-6a-27a^2=(1-9a)(1+3a),\ \textit{Ans.}$$

Factor the following:

47. $a^2-3ax+2x^2$.

48. $x^2+5xy-66y^2$.

49. $1+13a+42a^2$.

50. $m^2-15mn+56n^2$.

51. $a^2-ab-56b^2$.

52. $a^2b^2+4abc-45c^2$.

53. $1-3x-10x^2$.

54. $a^4+15a^3+44a^2$.

55. $z^2-10xy^2z-39x^2y^4$.

56. $(a+b)^2+5(a+b)+4$.

57. $1-9a+8a^2$.

58. $b^4+9ab^2-52a^2$.

59. $(m-n)^2+(m-n)-2$.

60. $x^6-5x^4-50x^2$.

61. $a^2+8ab+12b^2$.

62. $1-13xy+40x^2y^2$.

63. $(a-b)^2-3(a-b)-4$.

64. $x^4y^4+8x^2y^2z-48z^2$.

115. If a quantity can be resolved into three equal factors, it is said to be a *perfect cube*, and one of the equal factors is called its *cube root.*

Thus, since $27a^6b^3$ equals $3a^2b \times 3a^2b \times 3a^2b$, it is a perfect cube, and $3a^2b$ is its cube root.

116. It is evident from the above that the cube root of a monomial may be found by *extracting the cube root of the coefficient and dividing the exponent of each letter by* 3.

Thus, the cube root of $125x^6y^9z^3$ is $5x^2y^3z$.

Case VI.

117. *When an expression is the sum or difference of two perfect cubes.*

By Art. 98, the sum or difference of two perfect cubes is divisible by the sum or difference of their cube roots; and in either case, the quotient may be written by inspection by aid of the rules of Art. 98.

EXAMPLES.

1. Factor $a^3 + 1$.

The cube root of a^3 is a, and of 1 is 1; hence, one factor is $a + 1$.

Dividing the expression by $a + 1$, we have the quotient $a^2 - a + 1$ (Art. 98). Hence,

$$a^3 + 1 = (a + 1)(a^2 - a + 1), \textit{ Ans.}$$

2. Factor $27x^3 - 64y^3$.

The cube root of $27x^3$ is $3x$, and of $64y^3$ is $4y$; hence, one factor is $3x - 4y$. By Art. 98, the other factor is $9x^2 + 12xy + 16y^2$. Hence,

$$27x^3 - 64y^3 = (3x - 4y)(9x^2 + 12xy + 16y^2), \textit{ Ans.}$$

Factor the following:

3. a^3+x^3.	8. a^6+b^6.	13. m^3-64n^6.
4. m^3-n^3.	9. x^6+1.	14. $64x^3-125$.
5. x^3-1.	10. $27x^3-1$.	15. $125a^3+27m^3$.
6. $a^3b^3+c^3$.	11. $8c^6-d^9$.	16. $64c^3d^9+27$.
7. $1-8x^3$.	12. $27+8a^3$.	17. $125-8a^3b^6$.

Case VII.

118. *When an expression is the sum or difference of two equal odd powers of two quantities.*

By Art. 100, the sum or difference of two equal odd powers is divisible by the sum or difference of the quantities; and in either case, the quotient may be written by inspection by aid of the laws of Art. 99.

EXAMPLES.

1. Factor a^5+b^5.

By Art. 100, one factor is $a+b$. Dividing the expression by $a+b$, the quotient is $a^4-a^3b+a^2b^2-ab^3+b^4$ (Art. 99). Hence,

$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4), \textit{ Ans.}$$

Factor the following:

2. a^5-b^5.	5. m^7+n^7.	8. $c^5-m^5n^5$.
3. x^5+1.	6. x^7-y^7.	9. $1+32n^5$.
4. $1-a^5$.	7. a^7-1.	10. $243x^5-y^5$.

11. x^7+128. 12. $32-243a^5$.

119. By applying one or more of the rules already given, an expression may often be separated into more than two factors.

1. Factor $2ax^3y^2 - 8axy^4$.

By Case I., $2ax^3y^2 - 8axy^4 = 2axy^2(x^2 - 4y^2)$.

Factoring the quantity in the parenthesis by Case IV.,

$$2ax^3y^2 - 8axy^4 = 2axy^2(x + 2y)(x - 2y), \textit{ Ans.}$$

2. Factor $m^6 - n^6$.

By Case IV., $m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$.

By Case VI., $m^3 + n^3 = (m + n)(m^2 - mn + n^2)$,

and $m^3 - n^3 = (m - n)(m^2 + mn + n^2)$.

Hence,

$$m^6 - n^6 = (m+n)(m-n)(m^2 - mn + n^2)(m^2 + mn + n^2), \textit{ Ans.}$$

3. Factor $x^8 - y^8$.

By Case IV.,

$$\begin{aligned} x^8 - y^8 &= (x^4 + y^4)(x^4 - y^4) \\ &= (x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^4 + y^4)(x^2 + y^2)(x + y)(x - y), \textit{ Ans.} \end{aligned}$$

MISCELLANEOUS EXAMPLES.

120. In factoring the following expressions, the common monomial factors should be first removed, as shown in Example 1 of the preceding article.

1. $6a^2x^3 - 6a^4x$.

2. $1 - 4x + 4x^2$.

3. $x^6 - 1$.

4. $a^2 + 9a + 18$.

5. $x^2 + ax + bx + ab$.

6. $m^2 - 7m - 8$.

7. $2x^5 + x$.

8. $5a^3 - 5$.

9. $a^5b^5 - c^5d^5$.

10. $x^4 - 16$.

11. $a^5 - a^4 + a^3 - a^2$.

12. $3x^2 + 27x + 42$.

13. $x^2 - (2y - 3z)^2$.

14. $a^2 + 20ab + 100b^2$.

15. $5a^2bc - 10ab^2c - 15abc^2$.

16. $3a^4 - 21a^3 + 30a^2$.

17. $x^3 + 8y^3z^3$.

18. $2a^5 - 2a$.

19. $1 - a^2 - b^2 + 2ab$.

20. $x^2 - 8x + 7$.

21. $1 + 12x + 27x^2$.

22. $18x^3y - 2xy^3$.

23. $x^9 - x^2$.

24. $4x^2y^4 + 28xy^2 + 49$.

25. $a^6 + 6a^3 - 40$.

26. $a^2 - 18ab - 40b^2$.

27. $2x^3y + 2xy^3 - 2xyz^2 + 4x^2y^2$.

28. $12m^3n - 18m^2n^2 + 24mn^3$.

29. $32a^4b + 4ab^4$.

30. $x^4 - 81$.

31. $y - y^9$.

32. $x^3 + 2x^2 - x - 2$.

33. $x^2 + 7x^3 - 30x^4$.

34. $(3x + y)^2 - (x - 2y)^2$.

35. $m^2x^6 - 8mx^3 - 65$.

36. $135x^5 - 5x^2$.

37. $2x^3y - 2x^2y^2 - 60xy^3$.

38. $80x^2y^5 - 5x^6y$.

39. $3a^3b + 18a^2b + 27ab$.

40. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.

41. $a^2bc - ac^2d - ab^2d + bcd^2$.

42. $a^2 - 14ab + 33b^2$.

43. $3x^6y + 3xy^6$.

44. $4m^4 - 20m^2n + 25n^2$.

45. $3a^3b + 3a^2b^2 - 6ab^3$.

46. $a^2x^2 - a^2y^2 - b^2x^2 + b^2y^2$.

47. $(a - 2b)^2 - 2(a - 2b) - 8$.

48. $100x^2y^4 - 81z^2$.

49. $a^6 - 64$.

50. $x^4 - (x - 6)^2$.

51. $(a^2 + 3a)^2 - 14(a^2 + 3a) + 40$.

52. $(4m + n)^2 - (2m - 3n)^2$.

53. $(a^2 - b^2 - c^2)^2 - 4b^2c^2$.

54. $1000 + 27m^6$.

55. $x^3 - x^2 - x + 1$.

56. $3(a^2 - b^2) - (a - b)^2$.

57. $a^2 + b^2 - c^2 - d^2 - 2ab - 2cd$.

58. $(x^2 + 4)^2 - 16x^2$.

59. $x^3 - y^3 - 3xy(x - y)$.

60. $(a^2 + a - 4)^2 - 4$.

IX. HIGHEST COMMON FACTOR.

121. A **Common Factor** of two or more quantities is a quantity which will divide each of them without a remainder.

Thus, $2xy^2$ is a common factor of $12x^3y^3$ and $20x^2y^4$.

122. A *prime quantity* is one which cannot be divided, without a remainder, by any integral quantity except itself or unity.

For example, a, b, and $a+c$ are prime quantities.

123. Two quantities are said to be *prime to each other* when they have no common factor except unity.

Thus, $2a$ and $3b^2$ are prime to each other.

124. The **Highest Common Factor** of two or more quantities is the product of all the prime factors common to those quantities.

It is evident from this definition that the highest common factor of two or more quantities is the expression of *highest degree* (Art. 33) which will divide each of them without a remainder.

Thus, the highest common factor of x^3y^3 and x^2y^4 is x^2y^3.

125. In determining the highest common factor of algebraic quantities, it is convenient to distinguish three cases.

CASE I.

126. *When the quantities are monomials.*

1. Find the H.C.F. of $42a^3b^2$, $70a^2bc$, and $98a^4b^3d^2$.

$$\begin{aligned}42a^3b^2 &= 2\cdot 3\cdot 7\cdot a^3b^2\\ 70a^2bc &= 2\cdot 5\cdot 7\cdot a^2bc\\ 98a^4b^3d^2 &= 2\cdot 7\cdot 7\cdot a^4b^3d^2\end{aligned}$$

Hence, the H.C.F. $= 2\cdot 7\cdot a^2b$ (Art. 124) $= 14a^2b$, *Ans.*

RULE.

To the highest common factor of the coefficients, annex the common letters, giving to each the lowest exponent with which it occurs in any of the given quantities.

EXAMPLES.

Find the highest common factors of the following:

2. a^3x^2, $7a^4x$.
3. $15cd^2$, $9c^2d$.
4. $54a^3b$, $90ac^2$.
5. $18mn^5$, $45m^2n$, $72m^3n^2$.
6. $112xy^3z^2$, $154x^2yz^3$.
7. $15a^2x$, $45a^3y^2$, $60a^4z^3$.
8. $108x^3y^2z^7$, $144xy^3z^4$, $120x^5y^4z^5$.
9. $96a^5b^4$, $120a^3b^5$, $168a^4b^6$
10. $51a^2m^4n$, $85a^3m^3x$, $119a^4m^2y^4$.

CASE II.

127. *When the quantities are polynomials which can be readily factored by inspection.*

EXAMPLES.

1. Find the H.C.F. of

$$5x^3y - 15x^2y \text{ and } 10x^3y + 40x^2y - 210xy.$$

By the methods of Chapter VIII.,

$$5x^3y - 15x^2y = 5x^2y(x-3)$$

$$\begin{aligned} 10x^3y + 40x^2y - 210xy &= 10xy(x^2+4x-21) \\ &= 10xy(x+7)(x-3). \end{aligned}$$

In this case the common factors are 5, x, y, and $x-3$ Hence, the H.C.F. $= 5xy(x-3)$, *Ans.*

2. Find the H.C.F. of

$$4x^2 - 4x + 1,\ 4x^2 - 1, \text{ and } 2ax - a - 2bx + b.$$

$$4x^2-4x+1=(2x-1)(2x-1)$$
$$4x^2-1=(2x+1)(2x-1)$$
$$2ax-a-2bx+b=\underline{(a-b)(2x-1)}$$

Hence, the H.C.F. $=2x-1$, *Ans.*

Find the highest common factors of the following:

3. $3ax^2-2a^2x$ and a^2x^2-3abx.

4. x^2-y^2 and x^3+y^3.

5. $9a^4-4b^2$ and $(3a^2-2b)^2$.

6. $2x^6-2x^2$ and $6x^3-6x$.

7. $3cx+21c-3dx-21d$ and $x^2-3x-70$.

8. $m^3n+2m^2n^2+mn^3$ and m^4n+mn^4.

9. $3x^3+9x^2-120x$ and $3ax^2-9ax-30a$.

10. $3xy-4y+3xz-4z$ and $9x^2-16$.

11. x^2-x-42, $x^2-4x-60$, and $x^2+12x+36$.

12. a^2-1, a^3+1, and a^2+2a+1.

13. $4x^2-12x+9$, $4x^2-9$, and $4m^2nx-6m^2n$.

14. x^3-x, x^3+9x^2-10x, and x^6-x.

15. a^3-8b^3, $a^2-ab-2b^2$, and $a^2-4ab+4b^2$.

16. $2x^3+2x^2-4x$, $3x^4+6x^3-9x^2$, and $4x^5-20x^4+16x^3$.

17. $8m^3-125$, $4m^2-25$, and $4m^2-20m+25$.

18. x^4-16, x^2-x-6, and $(x^2-4)^2$.

19. $3ax^6-3ax^5$, ax^3-9ax^2+8ax, and $2ax^5-2ax$.

20. a^2-b^2, $ab-b^2+ac-bc$, and $a^3-a^2b+ab^2-b^3$.

21. $12ax-3a+8cx-2c$, $16x^2-1$, and $16x^2-8x+1$.

Case III.

128. *When the quantities are polynomials which cannot be readily factored by inspection.*

The rule in Arithmetic for the H.C.F. of two numbers, is

Divide the greater number by the less; if there is a remainder, divide the divisor by it; and so on; continuing the operation until there is no remainder. Then the last divisor is the highest common factor required.

For example, required the H.C.F. of 169 and 546.

```
169)546(3
    507
     39)169(4
        156
         13)39(3
            39
```

Therefore 13 is the H.C.F. required.

129. We will now prove that a similar rule holds for the H.C.F. of two algebraic quantities.

Let A and B be two expressions, the degree of A being not lower than that of B. Suppose that B is contained in A p times with a remainder C; that C is contained in B q times with a remainder D; and that D is contained in C r times with no remainder. To prove that D is the H.C.F. of A and B.

The operation of division is shown as follows:

```
B) A (p
   pB
    C) B (q
       qC
        D) C (r
           rD
```

We will first prove that D is a common factor of A and B.

From the nature of subtraction, the minuend is equal to the sum of the subtrahend and remainder (Art. 59).

Hence,
$$A = pB + C \qquad (1)$$
$$B = qC + D \qquad (2)$$
$$C = rD$$

Substituting the value of C in (2), we have
$$B = qrD + D = D(qr + 1) \qquad (3)$$

Substituting the values of B and C in (1), we have
$$A = pD(qr + 1) + rD = D(pqr + p + r) \qquad (4)$$

From (3) and (4) we see that D is a common factor of A and B.

We will next prove that every common factor of A and B is a factor of D.

Let K be any common factor of A and B, such that $A = mK$, and $B = nK$. From the operation of division, we see that
$$C = A - pB \qquad (5)$$
$$D = B - qC \qquad (6)$$

Substituting the values of A and B in (5), we have
$$C = mK - pnK.$$

Substituting the values of B and C in (6), we have
$$D = nK - q(mK - pnK) = K(n - qm + pqn).$$

Hence K is a factor of D.

Therefore, since every common factor of A and B is a factor of D, and since D is itself a common factor of A and B, it follows that D is the highest common factor of A and B.

130. Hence, to find the H.C.F. of two algebraic expressions, A and B, of which the degree of A is not lower than that of B,

Divide A by B; if there is a remainder, divide the divisor by it; and continue thus to make the remainder the divisor, and the preceding divisor the dividend, until there is no remainder. Then the last divisor is the highest common factor required.

Note 1. Each division should be continued until the remainder is of a lower degree than the divisor.

Note 2. It is important to keep the work in the same order of powers of some common letter, as in ordinary division.

1. Find the H.C.F. of

$$18x^3-51x^2+13x+5 \text{ and } 6x^2-13x-5.$$

$$\begin{array}{rl}
6x^2-13x-5) & 18x^3-51x^2+13x+5\,(3x-2 \\
 & \underline{18x^3-39x^2-15x} \\
 & \quad -12x^2+28x+5 \\
 & \quad \underline{-12x^2+26x+10} \\
 & \qquad\qquad 2x-5)\,6x^2-13x-5\,(3x+1 \\
 & \qquad\qquad\qquad \underline{6x^2-15x} \\
 & \qquad\qquad\qquad\qquad 2x-5 \\
 & \qquad\qquad\qquad\qquad \underline{2x-5}
\end{array}$$

Hence, $2x-5$ is the H.C.F. required.

Note 3. Either of the given expressions may be divided by any quantity which is not a factor of the other, as such a quantity can evidently form no part of the highest common factor. Similarly, any remainder may be divided by a quantity which is not a common factor of the given expressions.

2. Find the H.C.F. of

$$6x^3-25x^2+14x \text{ and } 6ax^2+11ax-10a.$$

Dividing the first expression by x, and the second by a, we have

$$\begin{array}{rl}
6x^2-25x+14) & 6x^2+11x-10\,(1 \\
 & \underline{6x^2-25x+14} \\
 & \qquad 36x-24
\end{array}$$

Dividing the remainder by 12,

$$
\begin{array}{r}
3x-2)\,6x^2-25x+14\,(2x-7 \\
\underline{6x^2-4x} \\
-21x+14 \\
\underline{-21x+14}
\end{array}
$$

Hence, $3x-2$ is the H.C.F. required.

Note 4. If the first term of a remainder is negative, the sign of each term may be changed.

3. Find the H.C.F. of $2x^2-3x-2$ and $2x^2-5x-3$.

$$
\begin{array}{r}
2x^2-3x-2)\,2x^2-5x-3\,(1 \\
\underline{2x^2-3x-2} \\
-2x-1
\end{array}
$$

Changing the sign of each term of this remainder,

$$
\begin{array}{r}
2x+1)\,2x^2-3x-2\,(x-2 \\
\underline{2x^2+x} \\
-4x-2 \\
\underline{-4x-2}
\end{array}
$$

Hence, $2x+1$ is the H.C.F. required.

Note 5. If the first term of the dividend or of any remainder is not divisible by the first term of the divisor, it may be made so by multiplying the dividend or remainder by any quantity which is not a factor of the divisor.

4. Find the H.C.F. of

$$2x^3-7x^2+5x-6 \text{ and } 3x^3-7x^2-7x+3.$$

Since $3x^3$ is not divisible by $2x^3$, we multiply the second quantity by 2.

$$
\begin{array}{r}
2x^3-7x^2+5x-6)\,6x^3-14x^2-14x+6\,(3 \\
\underline{6x^3-21x^2+15x-18} \\
7x^2-29x+24
\end{array}
$$

Since $2x^3$ is not divisible by $7x^2$, we multiply each term of the new dividend by 7.

$$
\begin{array}{r|l}
7x^2-29x+24)\,14x^3-49x^2+35x-42\,(2x \\
\underline{14x^3-58x^2+48x} \\
9x^2-13x-42
\end{array}
$$

Multiplying this by 7 to make its first term divisible by $7x^2$,

$$
\begin{array}{r}
7x^2-29x+24)\,63x^2-91x-294\,(9 \\
\underline{63x^2-261x+216} \\
170x-510
\end{array}
$$

Dividing by 170,

$$
\begin{array}{r}
x-3)\,7x^2-29x+24\,(7x-8 \\
\underline{7x^2-21x} \\
-8x+24 \\
\underline{-8x+24}
\end{array}
$$

Hence, $x-3$ is the H.C.F. required.

Note 6. If the given quantities have a common factor which can be seen by inspection, remove it, and find the H.C.F. of the resulting expressions. This result, multiplied by the common factor, will give the H.C.F. of the given quantities.

5. Find the H.C.F. of

$$6x^3-ax^2-5a^2x \text{ and } 21x^3-26ax^2+5a^2x.$$

Removing the common factor x, we find the H.C.F. of $6x^2-ax-5a^2$ and $21x^2-26ax+5a^2$. Multiplying the latter by 2,

$$
\begin{array}{r}
6x^2-ax-5a^2)\,42x^2-52ax+10a^2\,(7 \\
\underline{42x^2-7ax-35a^2} \\
-45ax+45a^2
\end{array}
$$

Dividing by $-45a$,

$$
\begin{array}{r}
x-a)\,6x^2-ax-5a^2\,(6x+5a \\
\underline{6x^2-6ax} \\
5ax-5a^2 \\
\underline{5ax-5a^2}
\end{array}
$$

Multiplying $x-a$ by x, the common factor, we have $x(x-a)$ or x^2-ax as the H.C.F. of the given expressions.

EXAMPLES.

131. Find the highest common factors of the following:

1. x^2+x-6 and $2x^2-11x+14$.
2. $6x^2-7x-24$ and $12x^2+8x-15$.
3. $2a^2-5a+3$ and $4a^3-2a^2-9a+7$.
4. $24x^2+11ax-28a^2$ and $40x^2-51ax+14a^2$.
5. $8a^3-22a^2+5a$ and $6a^2b-23ab+20b$.
6. $x^3-5mx^2+4m^2x$ and $x^4-mx^3+3m^2x^2-3m^3x$.
7. $5m^2n^2+58mn^2+33n^2$ and $10m^3+31m^2-20m-21$.
8. $2a^4+3a^3x-9a^2x^2$ and $6a^3-17a^2x+14ax^2-3x^3$.
9. x^3-8 and $x^3-6x^2+11x-6$.
10. $2x^3-3x^2-x+1$ and $6x^3-x^2+3x-2$.
11. $8m^2-22mn+5n^2$ and $6m^4-29m^3n+43m^2n^2-20mn^3$.
12. $ax^3+2ax^2+ax+2a$ and $3x^5-12x^3-3x^2-6x$.
13. $ax^4-ax^3-2ax^2+2ax$ and $ax^5-3ax^4+2ax^3+ax^2-ax$.
14. $2x^4-2x^3+4x^2+2x+6$ and $3x^4+6x^3-3x-6$.
15. $a^4+a^3-6a^2+a+3$ and $a^4+2a^3-6a^2-a+2$.
16. $x^5-x^4-5x^3+2x^2+6x$ and $x^5+x^4-x^3-2x^2-2x$.
17. $15a^2x^3-20a^2x^2-65a^2x-30a^2$
    and $12bx^3+20bx^2-16bx-16b$.
18. $a^4+a^3x+a^2x^2+ax^3-4x^4$
    and $a^4+2a^3x+3a^2x^2+4ax^3-10x^4$.

19. $x^4+x^3+x^2-1$ and x^5+3x^4+2x.

20. $x^4-x^3y-3x^2y^2+5xy^3-6y^4$
and $3x^4-5x^3y-x^2y^2-7xy^3+10y^4$.

21. $2x^4-5x^3+5x^2-5x+3$ and $2x^4-7x^3+4x^2+5x-3$.

22. $3a^4-2a^3b+2a^2b^2-5ab^3-2b^4$
and $6a^4-a^3b+2a^2b^2-2ab^3-b^4$.

132. To find the H.C.F. of three or more quantities, find the H.C.F. of two of them; then of this result and the third quantity, and so on. The last divisor will be the H.C.F. of the given quantities.

EXAMPLES.

Find the highest common factors of the following:

1. $2x^2-5x-42$, $4x^2+8x-21$, and $6x^2+23x+7$.

2. $12x^2-28x-5$, $14x^2-39x+10$, and $10x^2-11x-35$.

3. $6m^2+7mn+2n^2$, $3m^3-7m^2n-12mn^2-4n^3$,
and $15m^2+4mn-4n^2$.

4. $6a^2+13a-5$, $6a^3+19a^2+8a-5$,
and $3a^3+2a^2+2a-1$.

5. x^3+3x^2-6x-8, x^3+5x^2+2x-8,
and $x^3-3x^2-16x+48$.

6. x^3-7x+6, $x^3+3x^2-16x+12$,
and x^3-5x^2+7x-3.

7. $2a^3-3a^2-5a+6$, $2a^3+3a^2-8a-12$,
and $2a^3-a^2-12a-9$.

X. LOWEST COMMON MULTIPLE.

133. A **Common Multiple** of two or more quantities is a quantity which can be divided by each of them without a remainder.

Hence, a common multiple of two or more quantities must contain all the prime factors of each of the quantities.

134. The **Lowest Common Multiple** of two or more quantities is the product of their different prime factors, each being taken the greatest number of times which it occurs in any one of the quantities.

It is evident from this definition that the lowest common multiple of two or more quantities is the expression of *lowest degree* which can be divided by each of them without a remainder.

Thus, the lowest common multiple of x^3y^2, y^5z, and x^2z^4 is $x^3y^5z^4$.

When quantities are prime to each other, their product is their lowest common multiple.

135. In determining the lowest common multiple of algebraic quantities, we may distinguish three cases.

Case I.

136. *When the quantities are monomials.*

1. Find the L.C.M. of $36a^3x$, $60a^2y^2$, and $84cx^3$.

$$36a^3x = 2 \cdot 2 \cdot 3 \cdot 3 \cdot a^3x$$
$$60a^2y^2 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot a^2y^2$$
$$84cx^3 = 2 \cdot 2 \cdot 3 \cdot 7 \cdot cx^3$$

Hence, the L.C.M. $= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdot a^3cx^3y^2$ (Art. 134)

$= 1260a^3cx^3y^2$, *Ans.*

RULE.

To the lowest common multiple of the coefficients, annex all the letters which occur in the given quantities, giving to each the highest exponent which it has in any of the quantities.

EXAMPLES.

Find the lowest common multiples of the following:

2. $6a^3b$, a^2b^2.
3. $10x^2y$, $12y^3z$.
4. $30m^2$, $27n^2$.
5. $6ab$, $10bc$, $14ca$.
6. a^5b^2, $9a^3b^4$, $12a^2b^3$.
7. $16x^2y$, $42y^3z$.
8. $8c^2d^3$, $10ac$, $18a^2d$.
9. $24m^3x^2$, $30n^2y$, $32xy^2$.
10. $36xy^2z^3$, $63x^3yz^2$, $28x^2y^3z$.
11. $40a^2bd^3$, $90ac^3d^4$, $54b^3cd^2$.

CASE II.

137. *When the quantities are polynomials which can be readily factored by inspection.*

1. Find the L.C.M. of x^2+x-6, x^2-4x+4, and x^3-9x.

$$\begin{aligned} x^2+x-6 &= (x+3)(x-2) \\ x^2-4x+4 &= (x-2)^2 \\ x^3-9x &= x(x+3)(x-3) \end{aligned}$$

Hence, the L.C.M. $= x(x-2)^2(x+3)(x-3)$, (Art. 134)
$= x(x-2)^2(x^2-9)$, *Ans.*

EXAMPLES.

Find the lowest common multiples of the following:

2. x^2-y^2 and $xy-y^2$.
3. x^2-1 and x^2-7x-8.

4. $8a^2b + 8ab^2$ and $6a - 6b$.

5. $m^2 - n^2$ and $m^3 - n^3$.

6. $a - b$ and $a^2 - 4ab + 3b^2$.

7. $x^2 - 2xy + y^2$ and $x^3y - xy^3$.

8. $2a^2 + 2ab$, $3ab - 3b^2$, and $4a^2c - 4b^2c$.

9. $x^2 + 2ax - 35a^2$ and $x^2 - 2ax - 15a^2$.

10. $mn + n^2$, $mn - n^2$, and $m^2 - n^2$.

11. $ax - 2a + bx - 2b$ and $a^2 - 2ab - 3b^2$.

12. $ax^2 + a^2x$, $x^2 - a^2$, and $x^3 - a^3$.

13. $8(a^2 - b^2)$, $6(a + b)^2$, and $12(a - b)^2$.

14. $x^3 - 10x^2 + 21x$ and $ax^2 + 5ax - 24a$.

15. $x^2 - 1$, $x^2 - 2x + 1$, and $x^2 + 2x + 1$.

16. $2 - 2x^2$, $4 - 4x$, $8 + 8x$, and $12 + 12x^2$.

17. $x^2 + 5x + 4$, $x^2 + 2x - 8$, and $x^2 + 7x + 12$.

18. $a(x - b)(x - c)$, $b(x - c)(x - a)$, and $c(x - a)(x - b)$

19. $(2m - 1)^2$, $4m^2 - 1$, and $8m^3 - 1$.

20. $a^2 + a$, $a^4 - a^2$, and $a^6 + a^3$.

21. $a^2 - 4a + 3$, $a^2 + a - 12$, and $a^2 - a - 20$.

22. $1 - x^4$, $1 + 2x^2 + x^4$, and $1 - 2x^2 + x^4$.

23. $(a + b)^2 - c^2$ and $(a - c)^2 - b^2$.

24. $ax - ay - bx + by$, $(x - y)^2$, and $3a^2b - 3ab^2$.

25. $9x^3 + 12x^2 + 4x$, $18ax^4 - 12ax^3 + 8ax^2$,
and $27x^3 + 8$.

26. $x^2 - y^2 - z^2 + 2yz$ and $x^2 - y^2 + z^2 + 2xz$.

Case III.

138. *When the quantities are polynomials which cannot be readily factored by inspection.*

Let A and B be two expressions; let F be their highest common factor, and M their lowest common multiple. Suppose that $A = aF$ and $B = bF$; then,

$$A \times B = abF^2 \qquad (1)$$

Since a and b can have no common factor, the L.C.M. of aF and bF is abF; that is, $M = abF$; whence,

$$F \times M = abF^2 \qquad (2)$$

From (1) and (2) we have $A \times B = F \times M$ (Art. 42, 7).

That is, *the product of any two quantities is equal to the product of their highest common factor and lowest common multiple.*

Hence, to find the L.C.M. of two quantities,

Divide their product by their highest common factor; or,

Divide one of the quantities by their highest common factor, and multiply the quotient by the other quantity.

139. 1. Find the L.C.M. of

$$6x^2 - 17x + 12 \text{ and } 12x^2 - 4x - 21.$$

$$\begin{array}{rl}
6x^2 - 17x + 12) & 12x^2 - 4x - 21(2 \\
 & \underline{12x^2 - 34x + 24} \\
 & 30x - 45 \\
 & 2x - 3)6x^2 - 17x + 12(3x - 4 \\
 & \qquad \underline{6x^2 - 9x} \\
 & \qquad\quad - 8x + 12 \\
 & \qquad\quad \underline{- 8x + 12}
\end{array}$$

That is, the H.C.F. of the quantities is $2x - 3$. Dividing $6x^2 - 17x + 12$ by $2x - 3$, the quotient is $3x - 4$.

Hence, the L.C.M. $= (3x - 4)(12x^2 - 4x - 21)$

$= 36x^3 - 60x^2 - 47x + 84$, *Ans.*

EXAMPLES.

Find the lowest common multiples of the following:

2. $2x^2+x-6$ and $4x^2-8x+3$.

3. $6x^2+13x-28$ and $12x^2-31x+20$.

4. $8x^2+30x+7$ and $12x^2-29x-8$.

5. $6x^3-8x^2-30x$ and $6ax^2+19ax+15a$.

6. $a^2-8ab+7b^2$ and $a^3-9a^2b+23ab^2-15b^3$.

7. $2m^2n-3mn-2n$ and $2m^4-6m^3+6m^2-8m+8$.

8. $6ax^2-a^2x-12a^3$ and $10ax^2-17a^2x+3a^3$.

9. a^3+a^2-8a-6 and $2a^3-5a^2-2a+2$.

10. $2x^3+x^2-x+3$ and $2x^3+5x^2-x-6$.

11. $a^3-2a^2b+2ab^2-b^3$ and $a^3+a^2b-ab^2-b^3$.

12. $x^4+2x^3+2x^2+x$ and $ax^3-2ax-a$.

13. $2x^4-11x^3+3x^2+10x$ and $3x^4-14x^3-6x^2+5x$.

14. x^4-x^3-8x+8 and x^4-8x^2+9x-2.

140. To find the L.C.M. of three or more quantities, find the L.C.M. of two of them; then of this result and the third quantity; and so on.

EXAMPLES.

Find the lowest common multiples of the following:

1. x^2-1, $2x^2-9x+7$, and $2x^2+3x-5$.

2. $3a^2-2a-1$, $6a^2-a-1$, and $9a^2-3a-2$.

3. $2x^2-5x+2$, $4x^2+4x-3$, and $10x^2-7x+1$.

4. $4x^2-6x-18$, $4x^3+4x^2-3x$, and $6x^4+5x^3-6x^2$.

5. $a^3-6a^2+11a-6$, $a^3-a^2-14a+24$,
and $a^3+a^2-17a+15$.

XI. FRACTIONS.

141. The expression $\frac{a}{b}$ signifies $a \div b$; in other words, $\frac{a}{b}$ denotes that a units are divided into b equal parts, and that *one* part is taken.

Or, what is the same thing, $\frac{a}{b}$ denotes that *one* unit is divided into b equal parts, and that a parts are taken.

142. The expression $\frac{a}{b}$ is called a **Fraction**; a is called the *numerator*, and b the *denominator*.

By Art. 141, the denominator shows into how many parts the unit is divided, and the numerator shows how many parts are taken.

The numerator and denominator are called the *terms* of the fraction.

143. An **Entire Quantity** or **Integer** is one which has no fractional part; as $2xy$, or $a+b$.

Every integer may be considered as a fraction whose denominator is unity; thus, $a = \frac{a}{1}$.

144. A **Mixed Quantity** is one having both entire and fractional parts; as $a + \frac{b}{2}$, or $x + \frac{a}{y+z}$.

GENERAL PRINCIPLES.

145. *If the numerator of a fraction be multiplied, or the denominator divided, by any quantity, the fraction is multiplied by that quantity.*

I. Let $\frac{a}{b}$ be any fraction. Multiplying its numerator by c, we have $\frac{ac}{b}$. To prove that $\frac{ac}{b}$ is c times $\frac{a}{b}$.

In each of these fractions the unit is divided into b equal parts; in the first case ac parts are taken, and in the second case a parts. Since c times as many parts are taken in $\frac{ac}{b}$ as in $\frac{a}{b}$, it follows that

$$\frac{ac}{b} = c \times \frac{a}{b}. \tag{1}$$

II. Let $\frac{a}{bc}$ be any fraction. Dividing its denominator by c, we have $\frac{a}{b}$. To prove that $\frac{a}{b}$ is c times $\frac{a}{bc}$.

In each of these fractions a parts are taken; but since in the first case the unit is divided into b equal parts, and in the second case into bc equal parts, the parts in $\frac{a}{b}$ will be c times as great as in $\frac{a}{bc}$. Hence,

$$\frac{a}{b} = c \times \frac{a}{bc}. \tag{2}$$

146. *If the numerator of a fraction be divided, or the denominator multiplied, by any quantity, the fraction is divided by that quantity.*

I. Let $\frac{ac}{b}$ be any fraction. Dividing its numerator by c, we have $\frac{a}{b}$. To prove that $\frac{a}{b}$ is $\frac{ac}{b}$ divided by c.

By Art. 145, (1), $\quad c \times \frac{a}{b} = \frac{ac}{b}.$

Whence it follows that $\quad \frac{a}{b} = \frac{ac}{b} \div c.$

II. Let $\frac{a}{b}$ be any fraction. Multiplying its denominator by c, we have $\frac{a}{bc}$. To prove that $\frac{a}{bc}$ is $\frac{a}{b}$ divided by c.

By Art. 145, (2), $$c \times \frac{a}{bc} = \frac{a}{b}.$$

Whence it follows that $$\frac{a}{bc} = \frac{a}{b} \div c.$$

147. *If the numerator and denominator of a fraction be both multiplied, or both divided, by the same quantity, the value of the fraction is not altered.*

For, by Arts. 145 and 146, multiplying the numerator multiplies the fraction, and multiplying the denominator divides it. Hence, the fraction is both multiplied and divided by the same quantity, and its value is not altered.

Similarly we may show that if both terms are divided by the same quantity, the value of the fraction is not altered.

TO REDUCE A FRACTION TO ITS LOWEST TERMS.

148. A fraction is in its *lowest terms* when its numerator and denominator are prime to each other.

Case I.

149. *When the numerator and denominator can be readily factored by inspection.*

Since dividing both numerator and denominator by the same quantity, or canceling equal factors in each, does not alter the value of the fraction (Art. 147), we have the following rule:

Resolve both numerator and denominator into their prime factors, and cancel all which are common to both.

1. Reduce $\frac{18a^3b^2c}{45a^2b^2x}$ to its lowest terms.

$$\frac{18a^3b^2c}{45a^2b^2x} = \frac{2\cdot 3\cdot 3\cdot a^3b^2c}{3\cdot 3\cdot 5\cdot a^2b^2x}.$$

Dividing both terms by $3\cdot 3\cdot a^2b^2$, we have $\frac{2ac}{5x}$, *Ans.*

2. Reduce $\frac{x^3-27}{x^2-2x-3}$ to its lowest terms.

$$\frac{x^3-27}{x^2-2x-3} = \frac{(x-3)(x^2+3x+9)}{(x-3)(x+1)} = \frac{x^2+3x+9}{x+1}, \textit{ Ans.}$$

Note. If all the factors of the numerator be removed by cancellation, unity (which is a factor of all algebraic expressions) remains to form a numerator.

If all the factors of the denominator be removed, the result is an entire quantity; this being a case of exact division.

EXAMPLES.

3. $\frac{x^4y^2z}{xy^2z^3}$.

4. $\frac{2a^2b^5c}{5a^3bc^3}$.

5. $\frac{12xy^2}{32x^3}$.

6. $\frac{32mn}{56m^4n^3}$.

7. $\frac{65x^2y^3z^4}{26x^4y^3z^2}$.

8. $\frac{54a^3c^2}{72a^2bc}$.

9. $\frac{15mxy^2}{75mx^2y^3}$.

10. $\frac{115c^3x^2y}{23c^2x^2}$.

11. $\frac{154m^2x^3}{88m^3xy^2}$.

12. $\frac{2a^2cd+2abcd}{6a^2xy+6abxy}$.

13. $\frac{3x^5-6x^4y}{6x^2y^2-12xy^3}$.

14. $\frac{2x^3y-6x^2y}{x^2-8x+15}$.

15. $\frac{a^2-2a-15}{a^2+10a+21}$.

16. $\frac{6a^3b+3a^2b^2}{3a^2b^2+6ab^3}$.

17. $\frac{4c^2-20c+25}{4c^3-25c}$.

18. $\frac{m^2-10m+16}{m^2+m-72}$.

19. $\frac{9an^2-4a}{9bn^2-12bn+4b}$.

20. $\dfrac{a^2-4b^2}{a^2+ab-6b^2}$.

21. $\dfrac{8x^3+y^3}{4x^3-2x^2y+xy^2}$.

22. $\dfrac{ac-ad-bc+bd}{a^3-b^3}$.

23. $\dfrac{ax^2-4a}{x^3-9x^2+14x}$.

24. $\dfrac{27y^3-125}{9y^2-30y+25}$.

25. $\dfrac{x^3-x^2+2x-2}{2x^3+x^2+4x+2}$.

26. $\dfrac{x^2-4x+16}{ax^4+64ax}$.

27. $\dfrac{a^2-(b+c)^2}{(a-b)^2-c^2}$.

28. $\dfrac{(x^2-4)(x^2-3x+2)}{(x^2-4x+4)(x^2+x-2)}$.

29. $\dfrac{(a-b)^2-(c-d)^2}{(a-c)^2-(b-d)^2}$.

Case II.

150. *When the numerator and denominator cannot be readily factored by inspection.*

Since the H.C.F. of two quantities is the product of their common prime factors, we have the following rule:

Divide both numerator and denominator by their highest common factor.

EXAMPLES.

1. Reduce $\dfrac{2a^2-5a+3}{6a^2-a-12}$ to its lowest terms.

By the rule of Art. 130, the H.C.F. of $2a^2-5a+3$ and $6a^2-a-12$ is $2a-3$. Dividing the numerator by $2a-3$, the quotient is $a-1$; and dividing the denominator, the quotient is $3a+4$. Hence,

$$\frac{2a^2-5a+3}{6a^2-a-12}=\frac{a-1}{3a+4}, \quad \textit{Ans.}$$

Reduce the following to their lowest terms:

2. $\dfrac{x^2-6x+5}{3x^2+4x-7}$.

3. $\dfrac{10a^2-a-21}{2a^2-7a+6}$.

4. $\dfrac{2m^2-5m+3}{12m^2-28m+15}$.

5. $\dfrac{x^2-2x-3}{x^3-2x^2-2x-3}$.

6. $\dfrac{12m^2+16mn-3n^2}{10m^2+mn-21n^2}$.

7. $\dfrac{x^3+x^2-3x-2}{x^3-4x^2+2x+3}$.

8. $\dfrac{6x^3-7x^2+5x-2}{2x^3+5x^2-2x+3}$.

9. $\dfrac{6y^3-19y^2+7y+12}{6y^3-25y^2+17y+20}$.

10. $\dfrac{a^3-3a^2+a+2}{2a^3-3a^2-a-2}$.

11. $\dfrac{x^3-4x^2y+4xy^2-y^3}{x^3-2x^2y+4xy^2-3y^3}$.

151. Since a fraction represents the quotient of its numerator divided by its denominator, it is positive when its terms have the same sign, and negative when they have different signs.

Thus, if $\dfrac{a}{b}=x$,

then $$\frac{-a}{-b}=x, \text{ and } \frac{-a}{b}=\frac{a}{-b}=-x.$$

152. It follows from Art. 151 that the fraction $\dfrac{a}{b}$ can be written in any one of the forms

$$\frac{-a}{-b},\ -\frac{-a}{b},\text{ or }-\frac{a}{-b}.$$

That is, *if the signs of both numerator and denominator are changed, the value of the fraction is not altered. But if the sign of either one is changed, the sign before the fraction is changed.*

153. If either numerator or denominator is a polynomial, care must be taken, on changing its sign, to change the sign of *each of its terms.*

Thus, the fraction $\frac{a-b}{c-d}$, by changing the signs of both numerator and denominator, can be written in the form $\frac{-(a-b)}{-(c-d)}$, or $\frac{b-a}{d-c}$ (Art. 67).

154. It follows from Art. 151 that the fraction $\frac{ab}{cd}$ can be written in any one of the forms

$$\frac{(-a)b}{c(-d)}, \quad \frac{(-a)(-b)}{cd}, \quad \frac{(-a)(-b)}{(-c)(-d)}, \text{ etc.};$$

or, $$-\frac{(-a)b}{cd}, \quad -\frac{ab}{(-c)d}, \quad -\frac{(-a)(-b)}{c(-d)}, \text{ etc.}$$

From which it appears that

If the terms of a fraction are composed of factors, the signs of any even number of factors may be changed without altering the value of the fraction. But if the signs of any odd number of factors are changed, the sign before the fraction is changed.

Thus, the fraction $\frac{a-b}{(x-y)(x-z)}$ can be written in any one of the forms

$$\frac{a-b}{(y-x)(z-x)}, \quad \frac{b-a}{(y-x)(x-z)}, \quad -\frac{b-a}{(y-x)(z-x)}, \text{ etc.}$$

TO REDUCE A FRACTION TO AN ENTIRE OR MIXED QUANTITY.

155. Since a fraction is an expression of division, we have the following rule:

Divide the numerator by the denominator.

1. Reduce $\frac{6x^2-15x-2}{3x}$ to a mixed quantity.

Dividing each term of the numerator by the denominator,

$$\frac{6x^2-15x-2}{3x}=\frac{6x^2}{3x}-\frac{15x}{3x}-\frac{2}{3x}=2x-5-\frac{2}{3x}, \text{ Ans.}$$

2. Reduce $\frac{8x^3-12x^2-9x+10}{4x^2-3}$ to a mixed quantity.

$$\begin{array}{r|l} 4x^2-3)\,8x^3-12x^2-9x+10 & (2x-3 \\ \underline{8x^3\qquad\quad -6x\qquad} & \\ -12x^2-3x\qquad & \\ \underline{-12x^2\qquad +9} & \\ -3x+1 & \end{array}$$

A remainder whose first term will not contain the first term of the divisor, may be written over the divisor in the form of a fraction, and added to the quotient. Thus, the result is

$$2x-3+\frac{-3x+1}{4x^2-3}.$$

Or, since the sign of each term of the numerator may be changed, if at the same time the sign before the fraction is changed (Art. 152), we have

$$\frac{8x^3-12x^2-9x+10}{4x^2-3}=2x-3-\frac{3x-1}{4x^2-3}, \text{ Ans.}$$

EXAMPLES.

Reduce the following to mixed quantities:

3. $\frac{5x^2-10x+4}{5x}$.

4. $\frac{6x^3-3x^2+9x-7}{3x}$.

5. $\frac{x^3+2y^3}{x+y}$.

6. $\frac{2x^2-41}{x-3}$.

7. $\frac{a^3-a^2-a-2}{a^2+a-1}$.

8. $\frac{12x^2-8x+7}{4x-1}$.

9. $\dfrac{a^4+b^4}{a+b}$.

10. $\dfrac{4m^3-16m^2n+29mn^2-22n^3}{2m-3n}$.

11. $\dfrac{2a^4-a^3-9a^2+14}{2a^2-a-3}$.

12. $\dfrac{x^3+2x^2+3x+4}{x^2+x+1}$.

13. $\dfrac{x^5-y^5}{x+y}$.

14. $\dfrac{6x^3-13x^2+6x-6}{3x^2-2x+1}$.

TO REDUCE A MIXED QUANTITY TO A FRACTIONAL FORM.

156. The operation being the converse of that of Art. 155, we have the following rule:

Multiply the integral part by the denominator; add the numerator to the product when the sign before the fraction is $+$, *and subtract it when the sign is* $-$; *and write the result over the denominator.*

1. Reduce $\dfrac{x-5}{2x-3}+x-2$ to a fractional form.

By the rule,

$$\frac{x-5}{2x-3}+x-2=\frac{x-5+(x-2)(2x-3)}{2x-3}$$

$$=\frac{x-5+2x^2-7x+6}{2x-3}$$

$$=\frac{2x^2-6x+1}{2x-3},\ \textit{Ans.}$$

2. Reduce $a+b-\dfrac{a^2-b^2-5}{a-b}$ to a fractional form.

$$a+b-\frac{a^2-b^2-5}{a-b}=\frac{(a+b)(a-b)-(a^2-b^2-5)}{a-b}$$

$$=\frac{a^2-b^2-a^2+b^2+5}{a-b}=\frac{5}{a-b},\ \textit{Ans.}$$

Note. If the numerator is a polynomial, it will be found convenient to enclose it in a parenthesis, when the sign before the fraction is $-$.

EXAMPLES.

Reduce the following to fractional forms:

3. $x+1+\dfrac{x+1}{x}$.

4. $x+1-\dfrac{4}{x+3}$.

5. $\dfrac{2m^2-3n^2}{3m+n}+m-n$.

6. $7x-3-\dfrac{53x-20}{8}$.

7. $1-\dfrac{m-n}{m+n}$.

8. $a+b-\dfrac{a^2+b^2}{a+b}$.

9. $\dfrac{2}{2x+1}+3x-2$.

10. $a^2-b^2+\dfrac{ab(a+b)}{a-b}$.

11. $\dfrac{x+y}{x-y}-1$.

12. $m-n+\dfrac{m^2+n^2}{m+n}$.

13. $a^2-ab+b^2-\dfrac{2b^3}{a+b}$.

14. $x^2-3x-\dfrac{2x(3-x)}{x-2}$.

15. $\dfrac{m^3+n^3}{m^2+mn+n^2}-(m-n)$.

16. $1+2x+4x^2+\dfrac{x^3+1}{2x-1}$.

17. $x-2y-\dfrac{x^3-8y^3}{x^2-4xy+4y^2}$.

18. $x^2-2x+3-\dfrac{x^3+13x-5}{x^2+3x-2}$.

TO REDUCE FRACTIONS TO THEIR LOWEST COMMON DENOMINATOR.

157. 1. Reduce $\dfrac{5cd}{3a^2b}$, $\dfrac{3mx}{2ab^2}$, and $\dfrac{3ny}{4a^3b}$ to equivalent fractions having the lowest common denominator.

The lowest common denominator is the lowest common multiple of $3a^2b$, $2ab^2$, and $4a^3b$, which is $12a^3b^2$.

By Art. 147, both terms of a fraction may be multiplied by the same quantity without altering its value. Hence,

Multiplying both terms of $\frac{5cd}{3a^2b}$ by $4ab$, we have $\frac{20abcd}{12a^3b^2}$.

Multiplying both terms of $\frac{3mx}{2ab^2}$ by $6a^2$, we have $\frac{18a^2mx}{12a^3b^2}$.

Multiplying both terms of $\frac{3ny}{4a^3b}$ by $3b$, we have $\frac{9bny}{12a^3b^2}$.

Therefore the required fractions are

$$\frac{20abcd}{12a^3b^2},\ \frac{18a^2mx}{12a^3b^2},\ \text{and}\ \frac{9bny}{12a^3b^2},\ \textit{Ans.}$$

It will be observed that the terms of each fraction are multiplied by a quantity which is obtained by dividing the lowest common denominator by its own denominator. Hence the following rule:

Find the lowest common multiple of the given denominators. Divide this by each denominator separately, multiply the corresponding numerators by the quotients, and write the results over the common denominator.

Note. Before applying the rule, each fraction should be in its lowest terms.

EXAMPLES.

Reduce the following to equivalent fractions having the lowest common denominator:

2. $\frac{3ab}{14},\ \frac{2ac}{21},\ \text{and}\ \frac{5bc}{6}.$

3 $\frac{2}{a^3x^2},\ \frac{3}{ax^3},\ \text{and}\ \frac{4}{a^2x}.$

4. $\frac{4c-1}{8ab^2}\ \text{and}\ \frac{3b-2}{12a^2c}.$

5. $\frac{5az}{6x^2y},\ \frac{3bx}{8y^2z},\ \text{and}\ \frac{7cy-m}{10xz^2}.$

6. $\dfrac{2a}{a^2+a-6}$ and $\dfrac{4a}{a^2-4}$.

7. $\dfrac{1}{x^2-1}$ and $\dfrac{1}{x^3-1}$.

8. m, $\dfrac{m^3}{mn-n^2}$, and $\dfrac{mn^2}{m^2-n^2}$.

9. $\dfrac{2}{a-b}$, $\dfrac{3}{a+b}$, and $\dfrac{4}{a^2+b^2}$.

10. $\dfrac{ay}{1-x}$, $\dfrac{ax^2}{(1-x)^2}$, and $\dfrac{xy^3}{(1-x)^3}$.

11. $\dfrac{ab}{am-bm+an-bn}$ and $\dfrac{m-n}{2a^2-2ab}$.

12. $\dfrac{x+3}{x^2-3x+2}$, $\dfrac{x+1}{x^2-5x+6}$, and $\dfrac{x+2}{x^2-4x+3}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

158. It follows from the definition of Art. 141 that

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}, \text{ and } \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}.$$

Hence the following

RULE.

To add fractions, reduce them, if necessary, to equivalent fractions having the lowest common denominator. Add the numerators of the resulting fractions, and write the sum over the common denominator.

To subtract one fraction from another, reduce them to equivalent fractions having the lowest common denominator. Subtract the numerator of the subtrahend from that of the minuend, and write the result over the common denominator.

Note. The final result should be reduced to its simplest form.

1. Required the sum of $\frac{4a-1}{4ac}$ and $\frac{3-5b^2}{6b^2c}$.

The lowest common denominator is $12ab^2c$. Multiplying the terms of the first fraction by $3b^2$, and of the second by $2a$, we have

$$\frac{4a-1}{4ac}+\frac{3-5b^2}{6b^2c}=\frac{12ab^2-3b^2}{12ab^2c}+\frac{6a-10ab^2}{12ab^2c}$$

$$=\frac{12ab^2-3b^2+6a-10ab^2}{12ab^2c}$$

$$=\frac{2ab^2-3b^2+6a}{12ab^2c}, \textit{ Ans.}$$

2. Subtract $\frac{4x-1}{2x}$ from $\frac{6a-2}{3a}$.

The L.C.D. is $6ax$. Hence,

$$\frac{6a-2}{3a}-\frac{4x-1}{2x}=\frac{12ax-4x}{6ax}-\frac{12ax-3a}{6ax}$$

$$=\frac{12ax-4x-(12ax-3a)}{6ax}$$

$$=\frac{12ax-4x-12ax+3a}{6ax}$$

$$=\frac{3a-4x}{6ax}, \textit{ Ans.}$$

Note. If a fraction, whose numerator is a polynomial, is preceded by a $-$ sign, care must be taken to change the sign of each term in the numerator before combining it with the others. It is convenient in such a case to enclose the numerator in a parenthesis, as shown in Ex. 2.

EXAMPLES.

Simplify the following:

3. $\frac{2x-5}{12}+\frac{3x+11}{18}$.

4. $\frac{3}{5ab^2}-\frac{1}{2a^2b}$.

5. $\frac{2a+3}{6}-\frac{3a+5}{8}$.

6. $\frac{m-2}{2mn}-\frac{2-3mn^2}{3m^2n^3}$.

7. $\frac{b-4a}{24a}+\frac{a+5b}{30b}$.

8. $\frac{a-b}{4}+\frac{2a+b}{6}-\frac{3a-b}{8}$.

9. $\frac{a^2+1}{3a^2}-\frac{6a^3+1}{12a^3}+\frac{b-2}{6b}$.

10. $\frac{2x-1}{12}+\frac{2x+3}{15}-\frac{6x+1}{20}$.

11. $\frac{m+2}{7}-\frac{m+2}{14}-\frac{m+3}{21}$.

12. $\frac{2}{3}-\frac{2x-1}{6x}-\frac{3x^2+1}{9x^2}$.

13. $\frac{x-2}{2}+\frac{3x+1}{3}-\frac{6x-5}{4}-\frac{3}{5}$.

14. $\frac{3a+1}{12a}-\frac{2b-1}{8b}+\frac{4c-1}{16c}-\frac{6d+1}{24d}$.

15. Simplify $\frac{1}{x+x^2}+\frac{1}{x-x^2}$.

The L.C.M. of $x+x^2$ and $x-x^2$ is $x(1+x)(1-x)$, or $x(1-x^2)$. Multiplying the terms of the first fraction by $1-x$, and of the second by $1+x$, we have

$$\frac{1}{x+x^2}+\frac{1}{x-x^2}=\frac{1-x}{x(1-x^2)}+\frac{1+x}{x(1-x^2)}$$

$$=\frac{1-x+1+x}{x(1-x^2)}$$

$$=\frac{2}{x(1-x^2)},\ \textit{Ans.}$$

16. Simplify $\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4ab}{a^2-b^2}$.

The L.C.D. is a^2-b^2. Hence,

$$\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4ab}{a^2-b^2}$$

$$=\frac{(a+b)^2}{a^2-b^2}-\frac{(a-b)^2}{a^2-b^2}-\frac{4ab}{a^2-b^2}$$

$$=\frac{(a+b)^2-(a-b)^2-4ab}{a^2-b^2}$$

$$=\frac{a^2+2ab+b^2-(a^2-2ab+b^2)-4ab}{a^2-b^2}$$

$$=\frac{a^2+2ab+b^2-a^2+2ab-b^2-4ab}{a^2-b^2}=0, \textit{ Ans.}$$

Simplify the following:

17. $\frac{1}{1+x}+\frac{1}{1-x}$.

18. $\frac{1}{x+2}+\frac{1}{3-x}$.

19. $\frac{1}{x+7}-\frac{1}{x+8}$.

20. $\frac{a}{a-b}-\frac{b}{a+b}$.

21. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

22. $\frac{x+y}{y}-\frac{2xy+x^2}{y(x+y)}$.

23. $\frac{1+x}{1-x}-\frac{1-x}{1+x}$.

24. $\frac{m+n}{(m-n)^2}+\frac{2m}{m^2-n^2}$.

25. $\frac{1}{a^2-4a+4}-\frac{1}{a^2+a-6}$.

26. $\frac{x}{x-y}+\frac{3x}{x+y}-\frac{2xy}{x^2-y^2}$.

27. $\frac{a}{a+b}+\frac{b}{a-b}+\frac{2ab}{a^2-b^2}$.

28. $\frac{m}{mn-n^2}-\frac{1}{m-n}-\frac{1}{n}$.

29. $\frac{x+y}{x-y}+\frac{x-y}{x+y}+2$.

30. $\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}$.

31. $\dfrac{1}{a+b}+\dfrac{1}{a-b}-\dfrac{2a}{a^2+b^2}.$ 32. $\dfrac{1}{1-x}-\dfrac{x}{(1-x)^2}-\dfrac{x^2-4x}{(1-x)^3}.$

33. $\dfrac{1}{ab-cd}-\dfrac{1}{ab+cd}-\dfrac{2cd}{a^2b^2-c^2d^2}.$

34. $\dfrac{x-3}{x-2}-\dfrac{x+1}{x+5}+\dfrac{x+13}{x^2+3x-10}.$

35. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}-\dfrac{(a-b)^2}{(x-a)(x-b)}.$

36. $\dfrac{1}{x(x+1)}-\dfrac{1}{x(x-1)}+\dfrac{x}{x^2-1}.$

37. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}.$

38. $\dfrac{1}{x-1}-\dfrac{x}{x^2-1}+\dfrac{3}{x^3-1}.$

39. $\dfrac{2x-6}{x^2+3x+2}-\dfrac{x+2}{x^2-2x-3}-\dfrac{x+1}{x^2-x-6}.$

40. $\dfrac{x-y}{(x+z)(y+z)}+\dfrac{y-z}{(x+y)(x+z)}-\dfrac{z-x}{(x+y)(y+z)}.$

In certain cases, the principles of Arts. 152 and 154 enable us to change the form of a fraction to one which is more convenient for the purposes of addition and subtraction.

41. Simplify $\dfrac{3}{a-b}+\dfrac{2b+a}{b^2-a^2}.$

Changing the signs of the terms in the denominator of the second fraction, and at the same time changing the sign before the fraction (Art. 152), we have

$$\frac{3}{a-b}-\frac{2b+a}{a^2-b^2}.$$

The L.C.D. is now $a^2 - b^2$. Hence

$$\frac{3}{a-b} - \frac{2b+a}{a^2-b^2} = \frac{3(a+b)}{a^2-b^2} - \frac{2b+a}{a^2-b^2}$$

$$= \frac{3(a+b)-(2b+a)}{a^2-b^2}$$

$$= \frac{3a+3b-2b-a}{a^2-b^2} = \frac{2a+b}{a^2-b^2}, \textit{ Ans.}$$

42. Simplify

$$\frac{1}{(x-y)(x-z)} - \frac{1}{(y-x)(y-z)} - \frac{1}{(z-x)(z-y)}.$$

By Art. 154, we may change the sign of the factor $y-x$ in the second denominator, at the same time changing the sign before the fraction; and we may change the signs of both factors of the third denominator. The expression then becomes

$$\frac{1}{(x-y)(x-z)} + \frac{1}{(x-y)(y-z)} - \frac{1}{(x-z)(y-z)}.$$

The L.C.D. is now $(x-y)(x-z)(y-z)$. Hence the result

$$= \frac{(y-z)+(x-z)-(x-y)}{(x-y)(x-z)(y-z)} = \frac{y-z+x-z-x+y}{(x-y)(x-z)(y-z)}$$

$$= \frac{2y-2z}{(x-y)(x-z)(y-z)} = \frac{2(y-z)}{(x-y)(x-z)(y-z)}$$

$$= \frac{2}{(x-y)(x-z)}, \textit{ Ans.}$$

Simplify the following:

43. $\dfrac{4}{a^2-ab} + \dfrac{3}{b^2-ab}.$

44. $\dfrac{5a+1}{3a-3} - \dfrac{3a-1}{2-2a}.$

45. $\dfrac{1}{3x-x^2} + \dfrac{1}{x^2-9}.$

46. $\dfrac{1}{m^2-mn} - \dfrac{1}{n^2-m^2}.$

47. $\frac{1}{(a-2)(x+2)}+\frac{1}{(2-a)(x+a)}.$

48. $\frac{a}{a+b}+\frac{a}{b-a}+\frac{2a^2}{a^2-b^2}.$

49. $\frac{x}{1+x}-\frac{x}{1-x}-\frac{x^2}{x^2-1}.$

50. $\frac{3}{2-x}+\frac{5}{x-3}+\frac{1}{x^2-5x+6}.$

51. $\frac{1}{(a-b)(b-c)}+\frac{1}{(b-a)(a-c)}-\frac{1}{(c-a)(c-b)}.$

52. $\frac{2}{(x-2)(x-3)}-\frac{3}{(3-x)(4-x)}-\frac{1}{(x-4)(2-x)}.$

MULTIPLICATION OF FRACTIONS.

159. Required the product of $\frac{a}{b}$ and $\frac{c}{d}$.

In Arithmetic, $\frac{2}{3}$ times $\frac{5}{7}$ signifies two-thirds of $\frac{5}{7}$.

Similarly, in Algebra, $\frac{a}{b}\times\frac{c}{d}$ signifies a $b$ths of $\frac{c}{d}$. That is, *we divide $\frac{c}{d}$ by b, and multiply the result by a.*

By Art. 146, II., $\frac{c}{d}\div b=\frac{c}{bd}.$

By Art. 145, I., $\frac{c}{bd}\times a=\frac{ac}{bd}.$

Hence, $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$

We have therefore the following rule for the multiplication of fractions:

Multiply the numerators together for the numerator of the product, and the denominators for its denominator.

Mixed quantities should be reduced to a fractional form before applying the rule.

Common factors in the numerators and denominators should be canceled before performing the multiplication.

EXAMPLES.

1. Multiply $\dfrac{10a^3y}{9bx^2}$ by $\dfrac{3b^4x^3}{4a^3y^2}$.

$$\frac{10a^3y}{9bx^2}\times\frac{3b^4x^3}{4a^3y^2}=\frac{10\cdot 3\cdot a^3b^4x^3y}{9\cdot 4\cdot a^3bx^2y^2}$$

$$=\frac{5b^3x}{6y},\ \textit{Ans.}$$

2. Multiply together $\dfrac{x^2-2x}{x^2-2x-3}$, $\dfrac{x^2-9}{x^2-x}$, and $\dfrac{x^2+x}{x^2+x-6}$.

$$\frac{x^2-2x}{x^2-2x-3}\times\frac{x^2-9}{x^2-x}\times\frac{x^2+x}{x^2+x-6}$$

$$=\frac{x(x-2)}{(x-3)(x+1)}\times\frac{(x+3)(x-3)}{x(x-1)}\times\frac{x(x+1)}{(x+3)(x-2)}$$

$$=\frac{x}{x-1},\ \textit{Ans.}$$

Multiply the following:

3. $\dfrac{5a^2bc}{12mn^2}$ and $3mn$.

4. $\dfrac{3abx^2}{5ay^2}$ and $\dfrac{5xy^2}{3abx^3}$.

5. $\dfrac{2a}{3b}$, $\dfrac{6c}{5a}$, and $\dfrac{5b}{8c}$.

6. $\dfrac{8x^2}{9y^3}$, $\dfrac{15y^2}{16z^3}$, and $\dfrac{3z^4}{10x^3}$.

7. $\frac{3ab^2}{4cd}$, $\frac{3ac^2}{2bd}$, and $\frac{8ad^2}{9bc}$.

8. $\frac{3m^3}{4x^2}$, $\frac{2n^4}{21m^2}$, and $\frac{7x^2}{5mn^3}$.

9. $\frac{3x^2-x}{5x}$ and $\frac{10}{2x^2-4x}$.

10. $\frac{x^2-16}{x^2+5x}$ and $\frac{x^2-25}{x^2-4x}$.

11. $\frac{a-b}{a^2+2ab}$ and $\frac{a^2-4b^2}{a^2-ab}$.

12. $\frac{a^2-2ab+b^2}{a+b}$ and $\frac{b}{ax-bx}$.

13. $\frac{xy+y^2}{x^3-y^3}$ and $\frac{x^3+x^2y+xy^2}{(x+y)^2}$.

14. $\frac{a^3-a^2+a}{x^2+2x+4}$ and $\frac{x^3-8}{a^3+1}$.

15. $1+\frac{4}{x}-\frac{5}{x^2}$ and $\frac{3x}{x^2+x-2}$.

16. $\frac{1-x^2}{1-y}$, $\frac{1-y^2}{x+x^2}$, and $\frac{1}{1-x}$.

17. $\frac{x^2+5xy+6y^2}{x^2-4xy-21y^2}$ and $\frac{x^2-7xy}{x^2-4y^2}$.

18. $\frac{x^2y-4y}{(x-y)^2-z^2}$ and $\frac{x^2-xy-xz}{xy+2y}$.

19. $\frac{x^3-y^3}{x^2-xy+y^2}$, $\frac{x^3+y^3}{x^2+xy+y^2}$, and $1+\frac{y}{x-y}$.

20. $\frac{a^2-(b-c)^2}{(a+c)^2-b^2}$ and $\frac{a^2-(b+c)^2}{(a-c)^2-b^2}$.

DIVISION OF FRACTIONS.

160. Required the quotient of $\frac{a}{b}$ divided by $\frac{c}{d}$.

By Art. 85, we are to find a quantity which, when multiplied by $\frac{c}{d}$, will produce $\frac{a}{b}$.

That quantity is evidently $\frac{ad}{bc}$; hence,

$$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}.$$

We observe that the quotient is obtained by multiplying the dividend, $\frac{a}{b}$, by $\frac{d}{c}$, which is the divisor inverted. We have then the following rule for the division of fractions:

Invert the divisor, and proceed as in multiplication.

Mixed quantities should be reduced to a fractional form before applying the rule.

If the divisor is an integer, it may be written in a fractional form, as explained in Art. 143.

EXAMPLES.

1. Divide $\frac{6a^2b}{5x^3y^4}$ by $\frac{9a^2b^3}{10x^2y^5}$.

By the rule,

$$\frac{6a^2b}{5x^3y^4} \div \frac{9a^2b^3}{10x^2y^5} = \frac{6a^2b}{5x^3y^4} \times \frac{10x^2y^5}{9a^2b^3} = \frac{4y}{3b^2x}, \textit{ Ans}$$

2. Divide $\frac{x^2-9}{15}$ by $\frac{x^2+2x-3}{5}$.

$$\frac{x^2-9}{15} \div \frac{x^2+2x-3}{5} = \frac{(x+3)(x-3)}{15} \times \frac{5}{(x+3)(x-1)}$$

$$= \frac{x-3}{3(x-1)}, \textit{ Ans}$$

Divide the following:

3. $\frac{7a^3b}{5m^2n^3}$ by $14ab^4$.

4. $\frac{18mx^3}{25ny^2}$ by $\frac{6m^2x^4}{5n^2y^5}$.

5. $\frac{1}{a^2+a-12}$ by $\frac{1}{a^2+3a-18}$.

6. $\frac{1}{4}-\frac{4}{x^2}$ by $\frac{x^2}{12}+\frac{x}{3}$.

7. $\dfrac{x^3-25x}{x^2+x-6}$ by $\dfrac{x^2-5x}{x^2-x-12}$.

8. $\dfrac{ab-b^2}{a^2+2ab+b^2}$ by $\dfrac{b^2}{a^2-b^2}$.

9. $\dfrac{m^3+n^3}{m^2-2mn+n^2}$ by $\dfrac{m^2+mn}{m-n}$.

10. $\dfrac{a+1}{a^2-3a}$ by $\dfrac{a^2-a-2}{a^2-a-6}$.

11. $9+\dfrac{5y^2}{x^2-y^2}$ by $3+\dfrac{5y}{x-y}$.

12. $\dfrac{a^4-8ab^3}{a^2-2ab-3b^2}$ by $\dfrac{a^3+2a^2b+4ab^2}{a-3b}$.

13. $\dfrac{2}{3y^2}-\dfrac{2}{xy}+\dfrac{3}{2x^2}$ by $\dfrac{2}{3y^2}-\dfrac{3}{2x^2}$.

COMPLEX FRACTIONS.

161. A **Complex Fraction** is one having a fraction in its numerator or denominator, or both.

It may be regarded as a case in division; its numerator answering to the dividend, and its denominator to the divisor.

EXAMPLES.

1. Reduce $\dfrac{a}{b-\dfrac{c}{d}}$ to its simplest form.

$$\frac{a}{b-\dfrac{c}{d}}=\frac{a}{\dfrac{bd-c}{d}}=a\times\frac{d}{bd-c}=\frac{ad}{bd-c},\ \textit{Ans}$$

It is often advantageous to simplify a complex fraction by *multiplying both numerator and denominator by the lowest common multiple of their denominators.*

2. Reduce $\dfrac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}$ to its simplest form.

The L.C.M. of $a+b$ and $a-b$ is $(a+b)(a-b)$. Multiplying each of the component fractions by $(a+b)(a-b)$, we have

$$\frac{a(a+b)-a(a-b)}{b(a+b)+a(a-b)}=\frac{a^2+ab-a^2+ab}{ab+b^2+a^2-ab}=\frac{2ab}{a^2+b^2}, \textit{ Ans.}$$

3. Reduce $\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}$ to its simplest form.

$$\frac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}}=\frac{1}{1+\dfrac{x}{x+1}}=\frac{x+1}{x+1+x}=\frac{x+1}{2x+1}, \textit{ Ans.}$$

Reduce the following to their simplest forms:

4. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

5. $\dfrac{\dfrac{1}{b}-\dfrac{1}{a}}{\dfrac{a}{b}-\dfrac{b}{a}}$.

6. $\dfrac{x^2+\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

7. $\dfrac{a-2+\dfrac{1}{a}}{1-\dfrac{1}{a}}$.

8. $\dfrac{\dfrac{x^2+4y^2}{y}-4x}{\dfrac{1}{y}-\dfrac{2}{x}}$.

9. $\dfrac{x-7+\dfrac{12}{x}}{x+3-\dfrac{18}{x}}$.

10. $$\frac{\dfrac{m^2}{n^3}+\dfrac{1}{m}}{\dfrac{m}{n^2}-\dfrac{m-n}{mn}}.$$

17. $$\frac{1+\dfrac{2x^2}{1-x^2}}{1-x^2+\dfrac{4x^2}{1-x^2}}.$$

11. $$\frac{x-1-\dfrac{12}{x+3}}{x-5+\dfrac{12}{x+3}}.$$

18. $$\frac{\dfrac{a+b}{c+d}+\dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d}-\dfrac{a-b}{c+d}}.$$

12. $$2-\frac{1}{3+\dfrac{1}{\dfrac{x}{2}-2}}.$$

19. $$\frac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}.$$

13. $$\frac{\dfrac{a}{b}-\dfrac{b^2}{a^2}}{\dfrac{a}{b}+1+\dfrac{b}{a}}.$$

20. $$\frac{\dfrac{x+2y}{x+y}+\dfrac{x}{y}}{\dfrac{x+2y}{y}-\dfrac{x}{x+y}}.$$

14. $$\frac{\dfrac{x^2}{y^2}-4+\dfrac{3y^2}{x^2}}{\dfrac{x}{y}-\dfrac{3y}{x}}.$$

21. $$\frac{x-3a+\dfrac{4a^2}{a+x}}{x-\dfrac{2a^2}{a+x}}.$$

15. $$\frac{\dfrac{1}{1-x}-\dfrac{1}{1+x}}{\dfrac{1}{1-x}+\dfrac{1}{1+x}}.$$

22. $$\frac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$$

16. $$\frac{1-\dfrac{2b-2c}{a+b-c}}{1+\dfrac{2c}{a-b-c}}.$$

23. $$\frac{\dfrac{m-n}{m+n}+\dfrac{m^2+n^2}{m^2-n^2}}{\dfrac{m^2}{m-n}+\dfrac{m^2n+n^3}{(m-n)^2}}.$$

MISCELLANEOUS EXAMPLES.

162. Reduce the following to their simplest forms:

1. $c+\frac{2b}{x}-\frac{a+bx+cx^2}{x^2}$.

2. $\frac{m^3+4m^2-5m}{3m^3-75m}$.

3. $\frac{x^2(1+x)^3-x^3(1+x)^2}{(1+x)^6}$.

4. $\frac{a^2-b^2}{(m+n)^2}\div\frac{a^2-ab}{bm+bn}$.

5. $\frac{1+2x^2}{2+2x^2}-\frac{2+x}{2+2x}$.

6. $\frac{10a^2+30ab+20b^2}{5a^3+10a^2b}$.

7. $\left(x+1+\frac{1}{x}\right)\left(x-1+\frac{1}{x}\right)$.

8. $\frac{1-ax+a(x+a)}{(1-ax)^2+(x+a)^2}$.

9. $\left(\frac{a^2}{b^2}-2+\frac{b^2}{a^2}\right)\div\left(\frac{a}{b}-\frac{b}{a}\right)$.

10. $\frac{b(b-ax)+a(a+bx)}{(b-ax)^2+(a+bx)^2}$.

11. $\frac{ax}{ax+b}-\frac{b}{ax-b}+\frac{ax(3b-ax)}{a^2x^2-b^2}$.

12. $\frac{3x-3x^2}{2+4x+2x^2}\times\frac{10x+10x^2}{9-18x+9x^2}$.

13. $\frac{6n^5-48n^2}{9n^5+18n^4+36n^3}$.

14. $x^2-2y^2-\frac{6y^3-x^2y+xy^2}{x-3y}$.

15. $\frac{a+b}{a-b}-\frac{a-b}{a+b}-\frac{4b^2}{a^2-b^2}$.

16. $\left(x^3+x+\frac{1}{x}+\frac{1}{x^3}\right)\left(x-\frac{1}{x}\right)$.

17. $\frac{2x-1}{2x^2-2x+1}-\frac{2x+1}{2x^2+2x+1}$.

18. $\dfrac{\frac{1}{a}}{1+\frac{x^2}{a^2}}+\dfrac{1}{2}\left(\dfrac{1}{x-a}-\dfrac{1}{x+a}\right).$

19. $\dfrac{\frac{x}{x-y}-\frac{y}{x+y}}{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2-y^2}}.$ 20. $\dfrac{x^3-9x^2+26x-24}{x^3-12x^2+47x-60}.$

21. $a^2-3ab-2b^2-\dfrac{b^2(7a+6b)}{a-3b}.$

22. $\dfrac{2x+y}{x+y}-1-\dfrac{y}{y-x}-\dfrac{x^2}{x^2-y^2}.$

23. $\dfrac{(x+y+z)^2+(x-y)^2+(y-z)^2+(z-x)^2}{x^2+y^2+z^2}.$

24. $\dfrac{1}{x-2}-\dfrac{4}{(x-2)^2}-\dfrac{8(1-x)}{(x-2)^3}.$

25. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}.$

26. $\dfrac{(4x+y)^2-(x-2y)^2}{(3x-4y)^2-(2x+3y)^2}.$

27. $\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}-\dfrac{(a+b+c)^2}{(a+b)(b+c)(c+a)}.$

28. $\dfrac{2(1-3x)}{(1+x)(1+9x)}-\dfrac{1-2x}{(1+x)(1+4x)}+\dfrac{2}{1+4x}.$

29. $\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{3x^2}{x^3+1}-\dfrac{3x^2}{x^3-1}.$

30. $\left(\dfrac{a+b}{a-b}+\dfrac{a^2+b^2}{a^2-b^2}\right)\div\left(\dfrac{a-b}{a+b}-\dfrac{a^3-b^3}{a^3+b^3}\right).$

XII. SIMPLE EQUATIONS.

163. An **Equation** is a statement of the equality of two expressions.

The *First Member* of an equation is the expression on the left of the sign of equality, and the *Second Member* is the expression on the right of that sign.

Thus, in the equation $2x-3=3x-5$, the first member is $2x-3$, and the second is $3x-5$.

The *sides* of an equation are its two members.

164. A **Numerical Equation** is one in which all the known quantities are represented by numbers; as

$$2x-3=3x-5.$$

165. A **Literal Equation** is one in which some or all the known quantities are represented by letters; as

$$2x+3a=bx-4.$$

166. An **Identical Equation** is one whose two members are equal, whatever values are given to the letters involved; as

$$x^2-a^2=(x+a)(x-a).$$

167. The **Degree** of an equation, in which there is but one unknown quantity, is denoted by the highest power of the unknown quantity in the equation. Thus,

$$\left.\begin{array}{r} 2x-3=3x-5 \\ \text{and} \quad a^2x=bc-d \end{array}\right\} \text{are equations of the } \textit{first degree.}$$

$3x^2-2x=65$ is an equation of the *second degree*, etc.

168. A **Simple Equation** is an equation of the first degree.

169. The **Root** of an equation containing but one unknown quantity, is the value of the unknown quantity; or, it is the value which, when put in place of the unknown quantity, makes the equation identical.

Thus, the equation $5x - 7 = 3x + 1$, when 4 is put in place of x, becomes $20 - 7 = 12 + 1$, which is identical. Hence the root of the equation, or the value of x, is 4.

Note. A simple equation has but one root; but it will be seen hereafter that an equation may have two or more roots.

170. The *solution* of an equation is the process of finding its roots.

A root is *verified*, or the equation *satisfied*, when, on substituting the value of the root in place of its symbol, the equation becomes identical.

171. The operations required in the solution of an equation are based upon the following general principle, which is derived from the axioms of Art. 42:

If the same operations be performed upon equal quantities, the results will be equal.

Hence,

Both members of an equation may be increased, diminished, multiplied, or divided by the same quantity, without destroying the equality.

TRANSPOSITION.

172. *Any term may be transposed from one side of an equation to the other by changing its sign.*

For, consider the equation $x + a = b$.

Subtracting a from both members (Art. 171), we have

$$x + a - a = b - a;$$

or, by Art. 26, $$x = b - a,$$

where $+a$ has been transposed to the second member by changing its sign.

Again, consider the equation $x - a = b$.

Adding a to both members (Art. 171), we have

$$x - a + a = b + a;$$

or, $$x = b + a.$$

where $-a$ has been transposed to the second member by changing its sign.

Note. If the same term appear in both members of an equation affected with the same sign, it may be suppressed.

173. *The signs of all the terms of an equation may be changed without destroying the equality.*

For, consider the equation $a - x = b - c$.

Transposing each term (Art. 172), we have

$$c - b = x - a;$$

or, $$x - a = c - b,$$

which is the same as the original equation with every sign changed.

SOLUTION OF SIMPLE EQUATIONS.

174. 1. Solve the equation $5x - 7 = 3x + 1$.

Transposing the unknown quantities to the first member, and the known quantities to the second, we have

$$5x - 3x = 7 + 1.$$

Uniting the similar terms, $2x = 8$.

Dividing both members by 2 (Art. 171),

$$x = 4, \textit{ Ans.}$$

Note. The result may be verified by substituting the value of x in the given equation, as shown in Art. 109.

We have then the following rule for the solution of a simple equation containing but one unknown quantity:

Transpose the unknown terms to the first member, and the known terms to the second.

Unite the similar terms, and divide both members by the coefficient of the unknown quantity.

EXAMPLES.

2. Solve the equation $14 - 5x = 19 + 3x$.

Transposing, $-5x - 3x = 19 - 14.$

Uniting terms, $-8x = 5.$

Dividing by -8, $x = -\frac{5}{8}$, *Ans.*

Note. To verify this result, put $x = -\frac{5}{8}$ in the given equation. Then,

$$14 - 5\left(-\frac{5}{8}\right) = 19 + 3\left(-\frac{5}{8}\right)$$

Or, $$14 + \frac{25}{8} = 19 - \frac{15}{8}$$

Or, $$\frac{137}{8} = \frac{137}{8};$$ which is identical.

Solve the following equations:

3. $8x = 5x + 42.$

4. $7x + 5 = -30.$

5. $7x + 5 = x + 23.$

6. $9x + 7 = 3x - 11.$

7. $3x - 8 = 5x + 8.$

8. $5 - 6x = 1 - 4x.$

9. $5x + 14 = 17 - 3x.$

10. $3x - 31 = 11x - 16.$

11. $18 - 7x = 18x - 7.$

12. $27 + 10x = 13x + 23.$

13. $19x - 11 = 15 + 6x.$

14. $32x - 15 = 7 + 65x.$

15. $13x - 81 = 5x - 31x - 159.$

16. $12x - 20x + 13 = 9x - 259.$

17. Solve the equation

$$(2x-3)^2 - x(x+1) = 3(x-2)(x+7) - 5.$$

Performing the operations indicated, we have

$$4x^2 - 12x + 9 - x^2 - x = 3x^2 + 15x - 42 - 5.$$

Transposing,

$$4x^2 - 12x - x^2 - x - 3x^2 - 15x = -42 - 5 - 9.$$

Uniting terms, $-28x = -56.$

Dividing by -28, $x = 2$, *Ans.*

Solve the following equations:

18. $3 + 2(2x+3) = 2x - 3(2x+1).$

19. $2x - (4x-1) = 5x - (x-1).$

20. $7(x-2) - 5(x+3) = 3(2x-5) - 6(4x-1).$

21. $3(3x+5) - 2(5x-3) = 13 - (5x-16).$

22. $(2x-1)(3x+2) = (3x-5)(2x+20).$

23. $(5-6x)(2x-1) = (3x+3)(13-4x).$

24. $(x-3)^2 - (5-x)^2 = -4x.$

25. $(2x-1)^2 - 3(x-2) + 5(3x-2) - (5-2x)^2 = 0.$

26. $2(x-2)^2 - 3(x-1)^2 + x^2 = 1.$

27. $(x-1)(x-2)(x+4) = (x+2)(x+3)(x-4).$

28. $5(7+3x) - (2x-3)(1-2x) - (2x-3)^2 - (5+x) = 0.$

29. $(5x-1)^2 - (3x+2)^2 - (4x-3)^2 + 4 = 0.$

30. $(2x+1)^3 + (2x-1)^3 = 16x(x^2-4) - 228.$

SOLUTION OF EQUATIONS CONTAINING FRACTIONS.

175. 1. Solve the equation $\frac{2x}{3}-\frac{5}{4}=\frac{5x}{6}-\frac{9}{8}$.

The L.C.M. of 3, 4, 6, and 8 is 24. Multiplying each term of the equation by 24, we have

$$16x-30 \quad =20x-27$$

$$16x-20x=30 \quad -27$$

$$-4x=3$$

$$x=-\frac{3}{4},\ \textit{Ans.}$$

We have then the following rule for clearing an equation of fractions:

Multiply each term by the lowest common multiple of the denominators.

EXAMPLES.

Solve the following equations:

2. $x+\frac{x}{2}+\frac{x}{3}=-11.$

3. $\frac{3x}{4}-\frac{5x}{6}+\frac{1}{18}=0.$

4. $2x-\frac{3x}{4}=\frac{13}{14}-\frac{x}{7}.$

5. $\frac{7x}{4}-7=\frac{5x}{3}-\frac{9x}{4}.$

6. $\frac{1}{6}+\frac{1}{2x}=\frac{1}{4}+\frac{1}{12x}.$

7. $\frac{2x}{5}-\frac{9x}{20}-\frac{7x}{10}=\frac{5}{4}.$

8. $\frac{3}{x}-\frac{5}{2x}=7-\frac{3}{2x}.$

9. $\frac{x}{2}+\frac{11}{6}-\frac{x}{3}=\frac{x}{6}-\frac{3x}{4}.$

10. $x-\frac{x}{7}+20=\frac{x}{2}+\frac{x}{4}+26.$

11. $\frac{3}{x}-\frac{7}{2x}=\frac{7}{12}-\frac{5}{3x}.$

12. Solve the equation $\frac{3x-1}{4}-\frac{4x-5}{5}=4+\frac{7x+5}{10}$.

Multiplying through by 20, the L.C.M. of 4, 5, and 10,

$$15x-5-(16x-20)=80+14x+10$$
$$15x-5-16x+20=80+14x+10$$
$$15x-16x-14x=80+10+5-20$$
$$-15x=75$$
$$x=-5,\ \textit{Ans.}$$

Note. If a fraction whose numerator is a polynomial is preceded by a $-$ sign, care must be taken to change the sign of each term of the numerator when the denominator is removed. It is convenient, in such a case, to enclose the numerator in a parenthesis, as shown in the above example.

13. $3x+\frac{5x+3}{7}=\frac{7x}{2}$.

14. $x-\frac{2x+1}{5}=5x-\frac{5}{3}$.

15. $7x-\frac{11x-3}{4}=3x+7$.

16. $4x-\frac{2x-3}{3}+\frac{1}{2}(x-9)=5x$.

17. $x-(3x-4)-\frac{5-2x}{4}=2$.

18. $\frac{2x}{21}=x-7+\frac{x+3}{15}$.

19. $\frac{x+1}{2}-\frac{x+4}{5}=\frac{x-4}{7}$.

20. $2-\frac{7x-1}{6}=3x-\frac{19x+3}{4}$.

21. $\frac{5x-2}{3}-\frac{3x+4}{4}-\frac{7x+2}{6}=\frac{x-10}{2}$.

22. $\frac{1}{2}(x+1)-\frac{2x-5}{5}=\frac{11x+5}{10}-\frac{x-13}{3}$.

23. $\dfrac{5x+1}{3}+\dfrac{17x+7}{9}-\dfrac{1}{2}(3x-1)=\dfrac{7x-1}{6}.$

24. $\dfrac{4+x}{7}=\dfrac{1}{2}(3x-2)-\dfrac{11x+2}{14}-\dfrac{1}{3}(2-9x).$

25. $\dfrac{2x+1}{3}=\dfrac{4x+5}{4}-\dfrac{8+x}{6}+\dfrac{2x+5}{8}.$

26. $\dfrac{5x-1}{2}-\dfrac{7-3x}{3x}=\dfrac{10x-3}{4}-\dfrac{3-5x}{2x}.$

27. $\dfrac{3x+7}{2}-\dfrac{4(x^2-2)}{3x}-\dfrac{x^3+16}{6x^2}=\dfrac{7}{2}.$

28. Solve the equation $\dfrac{2}{x-1}-\dfrac{3}{x+1}-\dfrac{1}{x^2-1}=0.$

Multiplying each term by x^2-1, the L.C.M. of the denominators,

$$2(x+1)-3(x-1)-1=0$$
$$2x+2-3x+3-1=0$$
$$2x-3x=-2-3+1$$
$$-x=-4$$
$$x=4,\ \textit{Ans.}$$

29. Solve the equation $\dfrac{6x+1}{15}-\dfrac{2x-4}{7x-16}=\dfrac{2x-1}{5}.$

Multiplying each term by 15,

$$6x+1-\frac{30x-60}{7x-16}=6x-3.$$

Transposing and uniting terms, $4=\dfrac{30x-60}{7x-16}.$

Multiplying by $7x-16$, $28x-64=30x-60$

$$-2x=4$$
$$x=-2,\ \textit{Ans.}$$

Note. If the denominators are partly monomial and partly polynomial, it is often advantageous to clear of fractions at first *partially*; multiplying by a quantity which will remove the *monomial* denominators.

Solve the following equations:

30. $\frac{1}{3x-7}-\frac{2}{3x+7}=0.$

31. $\frac{2x-1}{3x+4}=\frac{2x+7}{3x+2}.$

32. $\frac{6x^2-7x+5}{2x^2+5x-13}=3.$

33. $\frac{5x-2}{x(x-1)}=\frac{5x+7}{x^2-1}.$

34. $\frac{x}{3}-\frac{x^2-5x}{3x-7}=\frac{2}{3}.$

35. $\frac{(x+5)^2}{x-3}=\frac{5x+1}{5}.$

36. $\frac{1}{x+1}+\frac{2}{x+2}=\frac{3}{x+3}.$

37. $\frac{3x+2}{6}-\frac{2x-1}{3x-7}=\frac{x}{2}.$

38. $\frac{2}{x-2}-\frac{1}{x-3}=\frac{1}{x^2-5x+6}.$

39. $\frac{6x+7}{15}-\frac{2(x-1)}{7x-6}=\frac{2x+1}{5}.$

40. $\frac{3}{1-x}-\frac{2}{1+x}-\frac{1}{1-x^2}=0.$

41. $\frac{2x^2+3x}{2x+1}+\frac{1}{3x}=x+1.$

42. $2\left(\frac{x+1}{x+2}\right)+3\left(\frac{x+2}{x+1}\right)=5.$

43. $\frac{6}{3x+1}-\frac{1}{x+1}=\frac{2}{2x-1}.$

44. $\frac{x}{9}=\frac{x+1}{3}-\frac{7-2x^2}{1-9x}.$

45. $\frac{(x+1)^2}{(x+2)^2}=\frac{x-4}{x-2}.$

46. $\dfrac{2x^2-3x+2}{3x^2+x-1}=\dfrac{2x-3}{3x+1}$.

47. $\dfrac{x-1}{x-2}+\dfrac{x+1}{x+2}=\dfrac{2(x^2+4x+1)}{(x+2)^2}$.

48. $\dfrac{4x+3}{10}-\dfrac{12x-5}{5x-29}-\dfrac{2x-1}{5}=0$.

49. $\dfrac{x-1}{x-2}-\dfrac{x-2}{x-3}=\dfrac{x-3}{x-4}-\dfrac{x-4}{x-5}$.

SOLUTION OF LITERAL EQUATIONS.

176. **1.** Solve the equation $2ax-3b=x+c-3ax$.

Transposing and uniting terms, $5ax-x=3b+c$.

Factoring the first member, $x(5a-1)=3b+c$.

Dividing by $5a-1$, $x=\dfrac{3b+c}{5a-1}$, *Ans.*

2. Solve the equation $(b-cx)^2-(a-cx)^2=b(b-a)$.

Performing the operations indicated,

$$b^2-2bcx+c^2x^2-(a^2-2acx+c^2x^2)=b^2-ab$$
$$b^2-2bcx+c^2x^2-a^2+2acx-c^2x^2=b^2-ab$$
$$2acx-2bcx=a^2-ab$$

Factoring both members, $2cx(a-b)=a(a-b)$

Dividing by $2c(a-b)$, $x=\dfrac{a(a-b)}{2c(a-b)}$

$=\dfrac{a}{2c}$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $2ax + d = 3c - bx.$

4. $6bmx - 5an = 15am - 2bnx.$

5. $x + 1 = 2ax - a^2(x - 1).$

6. $\frac{a^2}{x} + \frac{b}{2} = \frac{4b^2}{x} + \frac{a}{4}.$

7. $(a^2 - 2x)^2 = (4x - 3a^2)(x + a^2).$

8. $(2m + 3x)(2m - 3x) = n^2 - (3x - n)^2.$

9. $\frac{x - a}{b} - \frac{x + b}{a} + 2 = 0.$

10. $(x - a - b)^2 - (x - a)(x - b) + ab = 0.$

11. $\frac{x}{x - a} - \frac{x + 2b}{x + a} = \frac{a^2 + b^2}{x^2 - a^2}.$

12. $\frac{(b - 3x)(c + 2x)}{2(x - c)(b - 3c - 3x)} = 1.$

13. $(x+a)^3 - (x-a)^3 - a(3x-a)(2x+a) = x(a+1) + 3.$

14. $\frac{(n^2 - x^2)(n + x)}{x + 2n} = -x^2 + nx + n^2.$

15. $(a - x)(b - x) - a(b + 1) = \frac{a^2}{b} + x^2.$

16. $\frac{x}{2a} - 3 + \frac{x}{4a^3} = \frac{x}{3a^2} - 2a(2 - 3a).$

17. $\frac{x}{2} + \frac{1 - 2ax}{2a} + \frac{2x - 1}{a^2} = 0.$

18. $\frac{x}{mn}-\frac{x+mn}{3n}=\frac{x}{3n}-(m-1).$

19. $\frac{x+2a}{x-a}+\frac{x-3a}{x+a}=2.$

20. $\frac{4x-a}{2x-a}-\frac{x+a}{x-a}=1.$

21. $\frac{x}{2}-\frac{a-bcx}{2bc}=\frac{x}{6c}-\frac{ac-4bx}{3bc}.$

22. $\frac{ax+b}{ax-b}-\frac{3b}{ax+b}=\frac{a^2x^2+b^2}{a^2x^2-b^2}.$

23. $\frac{ax-b}{ax+b}-\frac{bx-a}{bx+a}=\frac{a-b}{(ax+b)(bx+a)}.$

24. $\frac{x-n}{m}-\frac{x^2-mx-n^2}{mx-n^2}=1+\frac{n^2}{mx-n^2}.$

SOLUTION OF EQUATIONS INVOLVING DECIMALS.

177. 1. Solve the equation $.2x-.01-.03x=.113x+.161$

Changing the decimals into common fractions,

$$\frac{2x}{10}-\frac{1}{100}-\frac{3x}{100}=\frac{113x}{1000}+\frac{161}{1000}.$$

Clearing of fractions,

$$200x-10-30x=113x+161$$

$$57x=171$$

$$x=3,\ \textit{Ans.}$$

Or, we may solve the equation as follows:

Transposing, $.2x - .03x - .113x = .01 + .161.$

Uniting terms, $.057x = .171.$

Dividing by .057, $x = 3$, *Ans.*

EXAMPLES.

Solve the following equations:

2. $.23x - 2.05 = .02x - 1.882.$

3. $.001x - .32 = .09x - .2x - .653.$

4. $.3x - .02 - .003x = .7 - .06x - .006.$

5. $.3(1.2x - 5) = 14 + .05x.$

6. $.7(x + .13) = .03(4x - .1) + .5.$

7. $3.3x - \dfrac{.72x - .55}{.5} = .1x + 9.9.$

8. $4.25 - \dfrac{.2}{x} = \dfrac{17}{4} - \dfrac{1 - .1x}{x}.$

9. $\dfrac{.6x + .044}{.4} - \dfrac{.5x - .178}{.6} = .38.$

10. $\dfrac{2 - 3x}{1.5} + \dfrac{5x}{1.25} - \dfrac{2x - 3}{9} = \dfrac{x - 2}{1.8} + \dfrac{25}{9}.$

XIII. PROBLEMS.

LEADING TO SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY.

178. For the solution of a problem by Algebra no general rule can be given, as much must depend on the skill and ingenuity of the student. A few suggestions, however, may be found of service:

1. *Express the unknown quantity, or one of the unknown quantities, by one of the final letters of the alphabet.*

2. *From the given conditions, find expressions for the other unknown quantities, if any, in the problem.*

3. *Form an equation in accordance with the conditions of the problem.*

4. *Solve the equation thus formed.*

PROBLEMS.

179. 1. What number is that to which if four-sevenths of itself be added, the sum will equal twice the number diminished by 27?

Let $x =$ the number.

Then, $\frac{4x}{7} =$ four-sevenths of it,

and $2x =$ twice it.

By the conditions, $x + \frac{4x}{7} = 2x - 27$

$$7x + 4x = 14x - 189$$

$$-3x = -189.$$

Whence, $x = 63$, the number required.

2. A is three times as old as B, and eight years ago he was seven times as old as B. Required their ages at present.

Let $x = \text{B's age.}$
Then, $3x = \text{A's age.}$
Also, $x - 8 = \text{B's age 8 years ago,}$
and $3x - 8 = \text{A's age 8 years ago.}$

By the conditions, $3x - 8 = 7(x - 8)$

$$3x - 8 = 7x - 56$$
$$-4x = -48.$$

Whence, $x = 12$, B's age,
and $3x = 36$, A's age.

Note. In the above solution we say "Let $x =$ B's age," meaning "Let $x =$ the number of years in B's age." Abbreviations of this nature are often used in Algebra; but it should be remembered that they are in fact abbreviations, and that x can only represent an abstract number.

3. A had twice as much money as B; but, after giving B \$35, he had only one-third as much as B. How much had each at first?

Let $x =$ what B had at first.
Then, $2x =$ what A had at first.

After giving B \$35, A had left $2x - 35$ dollars, while B had $x + 35$ dollars. Then, by the conditions,

$$x + 35 = 3(2x - 35)$$
$$x + 35 = 6x - 105$$
$$-5x = -140.$$

Whence, $x = 28$, B's money at first,
and $2x = 56$, A's money at first.

4. What number is that whose double exceeds its half by 45?

5. Divide 34 into two parts such that four-sevenths of one part may be equal to two-fifths of the other.

6. What number exceeds the sum of its third, tenth, and twelfth parts by 58?

7. Divide 59 into two parts such that the sum of one-seventh the greater and one-third the less shall be equal to 13.

8. A is four times as old as B, and in 30 years he will be only twice as old as B. What are their ages?

9. A is 62 years of age, and B is 36. How many years is it since A was three times as old as B?

10. A had one-half as much money as B; but after B had given him \$42, he had four times as much as B. How much had each at first?

11. Divide 207 into two parts such that one-fourth the greater shall exceed two-sevenths the less by 3.

12. What two numbers are those whose difference is 3, and the difference of whose squares is 51?

13. A drover paid \$1428 for a lot of oxen and cows. For the oxen he paid \$55 each, and for the cows \$32 each; and he has twice as many cows as oxen. How many has he of each?

14. Divide 80 into two parts such that if the greater is taken from 62, and the less from 48, the remainders are equal.

15. A gentleman left an estate of \$1872 to be divided between his wife, three sons, and two daughters. The wife was to receive three times as much as either of the daughters, and each son one-half as much as each of the daughters. How much did each receive?

16. Divide \$70 between A, B, and C, so that A's share may be three-eighths of B's, and C's share two-ninths of A's.

17. In a garrison of 2744 men, there are $12\frac{1}{2}$ times as many infantry as cavalry, and twice as many cavalry as artillery. How many are there of each kind?

18. A is 34 years older than B; and he is as much above 50 as B is below 40. Required their ages.

19. A man travelled 3036 miles. He went four-sevenths as many miles on foot as by water, and two-fifths as many miles on horseback as by water. How many miles did he travel in each manner?

20. Divide a into two parts such that m times the first part shall be equal to n times the second.

Let $x =$ the first part.

Then, $a - x =$ the second part.

By the conditions, $mx = n(a - x)$.

Or, $mx + nx = an$.

Whence, $x = \frac{an}{m+n}$, the first part.

Therefore, $a - x = a - \frac{an}{m+n} = \frac{am}{m+n}$, the second part.

21. Divide a into two parts such that m times the first shall be equal to the second divided by n.

22. Find four consecutive numbers whose sum is 94.

23. Divide 43 into two parts such that one of them shall be three times as much above 20 as the other lacks of 17.

24. Divide \$47 between A, B, C, and D, so that A and B together may have \$27, A and C \$25, and A and D \$23.

25. If a certain number is increased by 15, one-half the result is as much below 80 as the number itself is above 100. Required the number.

26. Divide 205 into four parts such that the second is one-half of the first, the third one-third of the second, and the fourth one-fourth of the third.

27. Eleven years ago, A was 4 times as old as B, and in 13 years he will be only twice as old. Required their ages at present.

28. Find two consecutive numbers such that the difference of their squares added to three times the greater number exceeds the less number by 92.

29. What number is that, five-sixths of which as much exceeds 25 as one-ninth of it is below 9?

30. A is m times as old as B, and in a years he will be n times as old. Required their ages at present.

31. Divide a into three parts such that the first may be n times the second, and the second n times the third.

32. A can do a piece of work in 8 days which B can perform in 10 days. In how many days can it be done by both working together?

Let $x =$ the number of days required.

Then, $\frac{1}{x} =$ what both can do in one day.

Also, $\frac{1}{8} =$ what A can do in one day,

and $\frac{1}{10} =$ what B can do in one day.

By the conditions, $\frac{1}{8} + \frac{1}{10} = \frac{1}{x}.$

$$5x + 4x = 40$$

$$9x = 40.$$

Whence, $x = 4\frac{4}{9}$, the number of days required.

33. A can do a piece of work in 15 days, and B can do the same in 18 days. In how many days can it be done by both working together?

34. A can do a piece of work in $3\frac{3}{8}$ hours which B can do in $2\frac{3}{4}$ hours, and C in $2\frac{1}{5}$ hours. In how many hours can it be done by all working together?

35. The stones which pave a square court would just cover a rectangular area whose length is 6 yards longer, and breadth 4 yards shorter, than the side of the square. Required the area of the court.

36. A, B, and C found a sum of money. It was agreed that A should receive \$15 less than one-half, B \$13 more than one-fourth, and C the remainder, which was \$27. How much did A and B receive?

37. A can do a piece of work in a hours which B can do in b hours. In how many hours can it be done by both working together?

38. A vessel can be filled by three taps; by the first alone it can be filled in a minutes, by the second in b minutes, and by the third in c minutes. In what time will it be filled if all the taps are opened?

39. A sum of money, amounting to \$4.32, consists entirely of dimes and cents, there being in all 108 coins. How many are there of each kind?

Let $x =$ the number of dimes.

Then, $108 - x =$ the number of cents.

Also, $10x =$ the value of the dimes in *cents*.

By the conditions,

$$10x + 108 - x = 432$$

$$9x = 324.$$

Whence, $x = 36$, the number of dimes,

and $108 - x = 72$, the number of cents.

40. A man has \$4.04 in dollars, dimes, and cents. He has one-fifth as many cents as dimes, and twice as many cents as dollars. How many has he of each kind?

41. A man has 3 shillings 7 pence in two-penny pieces and farthings; and he has 19 more farthings than two-penny pieces. How many has he of each kind?

42. I bought a picture for a certain sum, and paid the same price for a frame. If the frame had cost \$1 less, and the picture 75 cents more, the price of the frame would have been only half that of the picture. Required the cost of the picture.

43. A laborer agreed to serve for 36 days on condition that for every day he worked he should receive \$1.25, and for every day he was absent he should forfeit 50 cents. At the end of the time he received \$17. How many days did he work, and how many was he absent?

44. A has \$105, and B \$83. After giving B a certain sum, A has only one-third as much money as B. How much was given to B?

45. A has a dollars, and B b dollars. After giving B a certain sum, A has c times as much money as B. How much was given to B?

46. A vessel can be emptied by three taps; by the first alone it can be emptied in 80 minutes, by the second in 200 minutes, and by the third in 5 hours. In what time will it be emptied if all the taps are opened?

47. The second digit of a number exceeds the first by 2; and if the number, increased by 6, be divided by the sum of its digits, the quotient is 5. Required the number.

Let $x =$ the first digit.
Then, $x + 2 =$ the second,
and $2x + 2 =$ the sum of the digits.

The number itself is equal to 10 times the first digit, plus the second, which is $10x + x + 2$, or $11x + 2$. Hence, by the conditions,

$$\frac{11x+2+6}{2x+2} = 5$$

$$11x + 8 = 10x + 10.$$

Whence, $x = 2$.
Therefore, $11x + 2 = 24$, the number required.

48. The first digit of a number exceeds the second by 4; and if the number be divided by the sum of its digits, the quotient is 7. Required the number.

49. The first digit of a number is three times the second; and if the number, increased by 3, be divided by the difference of its digits, the quotient is 16. Required the number.

50. A merchant has grain worth 9 shillings per bushel, and other grain worth 13 shillings per bushel. In what proportion must he mix 40 bushels, so that the mixture may be worth 10 shillings per bushel?

51. Gold is $19\frac{1}{4}$ times as heavy as water, and silver $10\frac{1}{2}$ times. A mixed mass weighs 4160 ounces, and displaces 250 ounces of water. How many ounces of each metal does it contain?

52. The second digit of a number exceeds the first by 3; and if the number, diminished by 9, be divided by the sum of its digits, the quotient is 3. Required the number.

53. Two persons, A and B, 63 miles apart, start at the same time and travel towards each other. A travels 4 miles an hour, and B 3 miles an hour. How far will each have travelled when they meet?

Let $x =$ the distance A travels.

Then, $63 - x =$ the distance B travels.

Also, $\frac{x}{4} =$ the time A takes to travel x mi.,

and $\frac{63 - x}{3} =$ the time B takes to travel $63 - x$ mi.

By the conditions, $\frac{x}{4} = \frac{63 - x}{3}$

$$3x = 252 - 4x$$

$$7x = 252.$$

Whence, $x = 36$, the distance A travels,

and $63 - x = 27$, the distance B travels.

54. A person has $4\frac{1}{4}$ hours at his disposal. How far can he ride in a coach which travels 5 miles an hour, so as to return home in time, walking back at the rate of $3\frac{1}{2}$ miles an hour?

55. A courier who travels a miles daily is followed after n days by another, who travels b miles daily. In how many days will the second overtake the first?

56. Two men, A and B, 26 miles apart, set out, B 30 minutes after A, and travel towards each other. A travels 3 miles an hour, and B 4 miles an hour. How far will each have travelled when they meet?

57. A capitalist invests $\frac{3}{8}$ of a certain sum in 5 per cent bonds, and the remainder in 6 per cent bonds; and finds that his annual income is $180. Required the amount in each kind of bond.

58. What principal at r per cent interest will amount to a dollars in t years?

59. In how many years will p dollars amount to a dollars, at r per cent interest?

60. Separate 41 into two parts such that one divided by the other may give 1 as a quotient and 5 as a remainder.

Let $x =$ the divisor.

Then, $41 - x =$ the dividend.

By the conditions, $\dfrac{41-x}{x} = 1 + \dfrac{5}{x}$

$$41 - x = x + 5$$

$$-2x = -36.$$

Whence, $x = 18$, the divisor,

and $41 - x = 23$, the dividend.

61. Separate 37 into two parts such that one divided by the other may give 3 as a quotient and 1 as a remainder.

62. Separate 113 into two parts such that one divided by the other may give 2 as a quotient and 20 as a remainder.

63. A general, arranging his men in a solid square, finds he has 21 men over. But attempting to add 1 man to each side of the square, he finds he wants 200 men to fill up the square. Required the number of men on a side at first, and the whole number of troops.

64. Separate a into two parts such that one divided by the other may give b as a quotient and c as a remainder.

65. The denominator of a fraction exceeds the numerator by 6; and if 8 is added to the denominator, the value of the fraction is $\frac{1}{3}$. Required the fraction.

66. The sum of the digits of a number is 6, and the number exceeds its first digit by 46. What is the number?

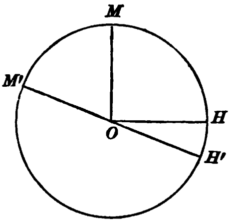
67. At what rate of interest will p dollars amount to a dollars in t years?

68. A man bought a picture for a certain price, and paid three-fourths the same amount for a frame. If the frame had cost \$2 less, and the picture 60 cents more, the price of the frame would have been one-third that of the picture. How much did each cost?

69. The denominator of a fraction exceeds the numerator by 1. If the denominator be increased by 2, the resulting fraction is less by unity than twice the original fraction. Required the fraction.

70. At what time between 3 and 4 o'clock are the hands of a watch opposite to each other?

Let OM and OH represent the positions of the minute and hour-hands at 3 o'clock, and OM' and OH' their positions when opposite to each other.



Let $x =$ the arc $MHH'M'$ over which the minute-hand has passed since 3 o'clock.

Then, $\frac{x}{12} =$ the arc HH' over which the hour hand has passed since 3 o'clock.

Also, the arc $MH = 15$ minute-spaces, and the arc $H'M' = 30$ minute-spaces.

Now, $\quad$ arc $MHH'M' =$ arc $MH +$ arc $H'M' +$ arc HH'.

That is, $$x = 15 + 30 + \frac{x}{12}$$

$$12x = 540 + x$$

$$11x = 540.$$

Whence, $$x = 49\tfrac{1}{11}.$$

Hence the required time is $49\frac{1}{11}$ minutes after 3 o'clock.

71. At what time between 7 and 8 are the hands of a watch opposite to each other?

72. At what time between 2 and 3 are the hands of a watch opposite to each other?

73. At what time between 5 and 6 are the hands of a watch together?

74. At what time between 1 and 2 are the hands of a watch together?

75. A woman sells half an egg more than half her eggs. Again she sells half an egg more than half her remaining eggs. A third time she does the same; and now she has sold all her eggs. How many had she at first?

76. A man has two kinds of money, dimes and half-dimes. If he is offered $1.35 for 20 coins, how many of each kind must he give?

77. A man has a hours at his disposal. How far can he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

78. At what time between 6 and 6.30 o'clock are the hands of a watch at right angles to each other?

79. At what times between 10 and 11 o'clock are the hands of a watch at right angles?

80. A banker has two kinds of money. It takes a pieces of the first kind to make a dollar, and b pieces of the second kind. If he is offered a dollar for c pieces, how many of each kind must he give?

81. A alone can perform a piece of work in 12 hours; A and C together can do it in 5 hours; and C's work is two-thirds of B's. The work must be completed at noon. A commences work at 5 A.M.; at what hour can he be relieved by B and C, and the work be just finished in time?

82. At what time between 4 and 5 is the minute-hand of a watch exactly 5 minutes in advance of the hour-hand?

83. A man buys a certain number of eggs at the rate of 3 for 10 cents. He sells one-third of them at the rate of 2 for 7 cents, and the remainder at the rate of 4 for 15 cents; and makes 16 cents by the transaction. How many eggs did he buy?

84. A merchant increases his capital annually by one-third of it, and at the end of each year sets aside $2700 for expenses. At the end of four years, after deducting the amount for expenses, he finds that his capital is reduced to $2980. What was his capital at first?

85. A man owns a harness valued at $25, a horse, and a carriage. The harness and carriage are together worth two-thirds the value of the horse, and the horse and harness are together worth $15 more than twice the value of the carriage. Required the value of the horse, and of the carriage.

86. Two men, A and B, 107 miles apart, set out at the same time and travel towards each other. A travels at the rate of 13 miles in 5 hours, and B at the rate of 11 miles in 4 hours. How far will each have travelled when they meet?

87. A mixture is made of a pounds of coffee at m cents a pound, b pounds at n cents, and c pounds at p cents. Required the cost per pound of the mixture.

88. A, B, and C together can do a piece of work in 6 days; B's work is one-half of A's, and C's work is two-thirds of B's. How many days will it take each working alone?

89. A and B start in business, A putting in $\frac{3}{2}$ as much capital as B. The first year, A gains $150, and B loses $\frac{1}{4}$ of his money. The next year, A loses $\frac{1}{4}$ of his money, and B gains $300; and they have now equal amounts. How much had each at first?

90. At what time between 9 and 10 is the hour-hand of a watch exactly one minute in advance of the minute-hand?

91. A and B together can do a piece of work in $1\frac{5}{7}$ days, A and C in $1\frac{7}{8}$ days, and B and C in $2\frac{2}{9}$ days. How many days will it take each working alone?

92. A man buys two pieces of cloth, one of which contains 3 yards more than the other. For the larger piece he pays at the rate of \$5 for 6 yards, and for the other at the rate of \$7 for 5 yards. He sells the whole at the rate of 3 yards for \$4, and makes \$8 by the transaction. How many yards were there in each piece?

93. A gentleman distributing some money among beggars, found that in order to give them a cents each, he should need b cents more. He therefore gave them c cents each, and had d cents left. Required the number of beggars.

94. A man let a certain sum for 3 years at 5 per cent compound interest; that is, at the end of each year there was added $\frac{1}{20}$ to the sum due. At the end of the third year there was due him \$2315.25. Required the sum let.

95. A man starts in business with \$4000, and adds to his capital annually one-fourth of it. At the end of each year he sets aside a fixed sum for expenses. At the end of three years, after deducting the fixed sum for expenses, his capital is reduced to \$2475. What are his annual expenses?

96. A man invests one-third of his money in $3\frac{1}{2}$ per cent bonds, two-fifths in 4 per cent bonds, and the balance in $4\frac{1}{2}$ per cent bonds. His income from the investments is \$595. What is the amount of his property?

97. At what time between 8 and 9 o'clock is the minute-hand of a watch exactly 35 minutes in advance of the hour-hand?

98. A fox is pursued by a greyhound, and has a start of 60 of her own leaps. The fox makes 3 leaps while the greyhound makes but 2; but the latter in 3 leaps goes as far as the former in 7. How many leaps does each make before the greyhound catches the fox?

XIV. SIMPLE EQUATIONS.

CONTAINING TWO UNKNOWN QUANTITIES.

180. If we have a simple equation containing *two* unknown quantities, as $x + y = 12$, it is impossible to determine the values of x and y *definitely;* because, if any value be assumed for one of the quantities, we can find a corresponding value for the other.

Thus, if $x = 9$, then $9 + y = 12$, or $y = 3$;

if $x = 8$, then $8 + y = 12$, or $y = 4$; etc.

Hence, any of the pairs of values,

$$x = 9,\ y = 3;\ x = 8,\ y = 4;\ \text{etc.},$$

will satisfy the given equation.

Similarly, the equation $x - y = 4$ is satisfied by any of the following pairs of values:

$$x = 9,\ y = 5;\ x = 8,\ y = 4;\ \text{etc.}$$

Equations of this kind are called *indeterminate.*

But suppose we are required to find a pair of values which will satisfy both $x + y = 12$ and $x - y = 4$ at the same time. It is evident by inspection that the values

$$x = 8,\ y = 4$$

satisfy both equations; and no other pair of values can be found which will satisfy both simultaneously.

181. Simultaneous Equations are such as are satisfied by the *same values* of their unknown quantities.

Independent Equations are such as cannot be made to assume the same form.

Thus, $x+y=9$ and $x-y=1$ are independent equations.

But $x+y=9$ and $2x+2y=18$ are not independent, since the first equation may be obtained from the second by dividing each term by 2.

182. It is evident from Art. 180 that two unknown quantities require for their determination *two* independent, simultaneous equations.

Two such equations may be solved by combining them so as to form a single equation containing but *one* unknown quantity. This operation is called **Elimination.**

183. There are three principal methods of elimination:

1. By Addition or Subtraction.
2. By Substitution.
3. By Comparison.

ELIMINATION BY ADDITION OR SUBTRACTION.

184. 1. Solve the equations $\begin{cases} 5x-3y=19 & (1) \\ 7x+4y=2 & (2) \end{cases}$

Multiplying (1) by 4, $\quad 20x-12y=76$

Multiplying (2) by 3, $\quad 21x+12y=6$

Adding these equations, $\quad 41x=82$

Whence, $\quad x=2$

Substituting the value of x in (1), $10-3y=19$

$$-3y=9$$

Whence, $\quad y=-3$

Ans. $x=2,\ y=-3.$

This solution is an example of elimination by *addition.*

2. Solve the equations $\begin{cases} 15x+8y=1 & (1) \\ 10x-7y=-24 & (2) \end{cases}$

$$
\begin{array}{lrl}
\text{Multiplying (1) by 2,} & 30x+16y= & 2 \quad (3) \\
\text{Multiplying (2) by 3,} & 30x-21y= & -72 \quad (4) \\
\textit{Subtracting (4) from (3),} & 37y= & 74 \\
 & y= & 2 \\
\text{Substituting this value in (2),} & 10x-14= & -24 \\
 & 10x= & -10 \\
 & x= & -1
\end{array}
$$

Ans. $x=-1$, $y=2$.

This solution is an example of elimination by *subtraction.*

RULE.

Multiply the given equations by such numbers as will make the coefficients of one of the unknown quantities equal. Add or subtract the resulting equations according as the equal coefficients have unlike or like signs.

Note. If the coefficients which are to be made equal are prime to each other, each may be used as the multiplier for the other equation. If they are not prime, such multipliers should be used as will produce their lowest common multiple.

Thus, in Ex. 1, to make the coefficients of y equal, we multiply (1) by 4, and (2) by 3. But in Ex. 2, to make the coefficients of x equal, since the L.C.M. of 15 and 10 is 30, we multiply (1) by 2, and (2) by 3.

EXAMPLES.

Solve the following by the method of addition or subtraction:

3. $\begin{cases} 7x+2y=31. \\ 3x-4y=23. \end{cases}$

4. $\begin{cases} 3x+7y=33. \\ x+2y=10. \end{cases}$

5. $\begin{cases} 2x-3y=4. \\ 6x-y=28. \end{cases}$

6. $\begin{cases} 7y-5x=-11. \\ 15x-14y=82. \end{cases}$

7. $\begin{cases} 2x-3y=-24. \\ 3x+2y=3. \end{cases}$

8. $\begin{cases} 9x-13y=76. \\ 15x+4y=101. \end{cases}$

9. $\begin{cases} 24x+13y=-27. \\ 36x+11y=-15. \end{cases}$

10. $\begin{cases} 15y-8x=12. \\ 25y+12x=1. \end{cases}$

11. $\begin{cases} 5x-7y=15. \\ 3x-5y=13. \end{cases}$

12. $\begin{cases} 7x-11y=-58. \\ 15x+8y=2. \end{cases}$

13. $\begin{cases} 11y-18x=2. \\ 24x-5y=-22. \end{cases}$

14. $\begin{cases} 24x-18y=-43. \\ 42x+30y=17. \end{cases}$

15. $\begin{cases} 11x-12y=-32. \\ 11y-12x=14. \end{cases}$

16. $\begin{cases} 9x-11y=24. \\ 10x+9y=-37. \end{cases}$

17. $\begin{cases} 12x+21y=-23. \\ 15x+28y=-30. \end{cases}$

ELIMINATION BY SUBSTITUTION.

185. 1. Solve the equations $\begin{cases} 7x-3y=-62 & (1) \\ 2y-5x=44 & (2) \end{cases}$

Transposing $5x$ in (2), $2y=5x+44$

Or, $y=\dfrac{5x+44}{2}$ (3)

Substituting this in (1), $7x-3\left(\dfrac{5x+44}{2}\right)=-62$

Or, $7x-\dfrac{15x+132}{2}=-62$

Clearing of fractions, $14x-15x-132=-124$

$-x=8$

Whence, $x=-8$

Substituting this value in (3), $y=\dfrac{-40+44}{2}=2$

Ans. $x=-8$, $y=2$.

RULE.

Find the value of one of the unknown quantities in terms of the other from one of the given equations, and substitute this value for that quantity in the other equation.

EXAMPLES.

Solve the following by the method of substitution:

2. $\begin{cases} x + y = 7. \\ 3x + 2y = 19. \end{cases}$

3. $\begin{cases} 3x - y = 10. \\ x + 4y = -1. \end{cases}$

4. $\begin{cases} 3x - 4y = 2. \\ 2x - 5y = 6. \end{cases}$

5. $\begin{cases} 7x - 2y = 8. \\ 8y - 5x = -9. \end{cases}$

6. $\begin{cases} 9x - 4y = -4. \\ 15x + 8y = -3. \end{cases}$

7. $\begin{cases} 2x - 7y = 8. \\ 4y - 9x = 19. \end{cases}$

8. $\begin{cases} 5x + 7y = -19. \\ 4x + 5y = -14. \end{cases}$

9. $\begin{cases} 10x - 7y = 9. \\ 4y - 15x = -7. \end{cases}$

10. $\begin{cases} 6x - 5y = -7. \\ 10x + 3y = 11. \end{cases}$

11. $\begin{cases} 9x + 2y = 15. \\ 4x + 7y = 3. \end{cases}$

12. $\begin{cases} 8x + 7y = -23. \\ 5y - 12x = -12. \end{cases}$

13. $\begin{cases} 7y - 3x = 139. \\ 2x + 5y = 91. \end{cases}$

ELIMINATION BY COMPARISON.

186. 1. Solve the equations $\begin{cases} 2x - 5y = -16 & (1) \\ 3x + 7y = 5 & (2) \end{cases}$

Transposing $-5y$ in (1), $2x = 5y - 16$

Or, $x = \dfrac{5y - 16}{2}$ (3)

Transposing $7y$ in (2), $3x = 5 - 7y$

Or, $x = \dfrac{5 - 7y}{3}$

Equating these values of x, $\frac{5y-16}{2}=\frac{5-7y}{3}$

Clearing of fractions, $15y-48=10-14y$

$$29y=58$$

$$y=2$$

Substituting this value in (3), $x=\frac{10-16}{2}=-3$

Ans. $x=-3,\ y=2.$

RULE.

Find the value of the same unknown quantity in terms of the other from each of the given equations, and place these values equal to each other.

EXAMPLES.

Solve the following by the method of comparison:

2. $\begin{cases} x-\ \ y=-1. \\ 3x+5y=\ 21. \end{cases}$

3. $\begin{cases} 6x+5y=-8. \\ 4x+3y=-5. \end{cases}$

4. $\begin{cases} 3x-5y=\ \ 25. \\ 7y-2x=-24. \end{cases}$

5. $\begin{cases} 3x-10y=-36. \\ 2x-\ \ 9y=-31. \end{cases}$

6. $\begin{cases} 3x-5y=51. \\ 2x+7y=\ 3. \end{cases}$

7. $\begin{cases} 7x+\ \ y=-3. \\ x+6y=\ 23. \end{cases}$

8. $\begin{cases} 5x+6y=\ \ 24. \\ 9y-8x=-26. \end{cases}$

9. $\begin{cases} 7x-\ \ 8y=-11. \\ x-12y=\ \ 12. \end{cases}$

10. $\begin{cases} 5x-12y=7. \\ 10x-\ \ 9y=4. \end{cases}$

11. $\begin{cases} 7y-12x=17. \\ 8x+11y=20. \end{cases}$

12. $\begin{cases} 7x+3y=6. \\ 11x+9y=8. \end{cases}$

13. $\begin{cases} 15x+6y=-7. \\ 8y-21x=\ 18. \end{cases}$

MISCELLANEOUS EXAMPLES.

187. Before applying either method of elimination, each of the given equations should be reduced to its simplest form.

1. Solve the equations

$$\begin{cases} \dfrac{7}{x+3} - \dfrac{3}{y+4} = 0 & (1) \\ x(y-2) - y(x-5) = -13 & (2) \end{cases}$$

From (1),

$$7y + 28 - 3x - 9 = 0, \text{ or } 7y - 3x = -19 \qquad (3)$$

From (2),

$$xy - 2x - xy + 5y = -13, \text{ or } 5y - 2x = -13 \qquad (4)$$

Multiplying (3) by 2, $\quad 14y - 6x = -38 \qquad (5)$

Multiplying (4) by 3, $\quad 15y - 6x = -39 \qquad (6)$

Subtracting (5) from (6), $\quad y = -1$

Substituting in (4), $\quad -5 - 2x = -13$

$$-2x = -8$$

$$x = 4.$$

Ans. $x = 4$, $y = -1$.

Solve the following:

2. $\begin{cases} 11y + 6x = 115. \\ \dfrac{2x}{3} - \dfrac{11y}{6} = -\dfrac{5}{2}. \end{cases}$

3. $\begin{cases} \dfrac{x}{3} + 3y = -46. \\ \dfrac{y}{3} + 3x = 66. \end{cases}$

4. $\begin{cases} \dfrac{3x}{2} - \dfrac{5y}{3} = -4. \\ \dfrac{x}{8} + \dfrac{y}{6} = -4. \end{cases}$

5. $\begin{cases} \dfrac{x}{3} - \dfrac{y}{4} = \dfrac{5}{6}. \\ \dfrac{x}{5} - \dfrac{y}{6} = \dfrac{47}{90}. \end{cases}$

6. $\begin{cases} .2x - .05y = .25. \\ .03x + .3y = .96. \end{cases}$

7. $\begin{cases} .5x + 2y = .01. \\ .11x + .3y = -.009. \end{cases}$

8. $\begin{cases} \dfrac{x+y}{2} - \dfrac{x-y}{3} = 8. \\[2ex] \dfrac{x+y}{3} + \dfrac{x-y}{4} = 11. \end{cases}$

9. $\begin{cases} 10 - \dfrac{2x+3y}{5} = \dfrac{y}{3}. \\[2ex] \dfrac{4y-3x}{6} = \dfrac{3x}{4} + 1. \end{cases}$

10. $\begin{cases} x(2y-3) = 2y(x+1). \\[2ex] \dfrac{3}{x-1} + \dfrac{5}{y+2} = 0. \end{cases}$

11. $\begin{cases} x(y-3) - y(x+4) = 22. \\ (y+1)(x-2) - (y+3)(x-4) = 6. \end{cases}$

12. $\begin{cases} \dfrac{x+y}{x-y} = \dfrac{5}{3}. \\[2ex] \dfrac{x+y+1}{x-y-1} = 7. \end{cases}$

15. $\begin{cases} \dfrac{2x+3y}{x+y+13} = -\dfrac{1}{2}. \\[2ex] \dfrac{5x}{3} - \dfrac{7y-2}{5} = 11. \end{cases}$

13. $\begin{cases} \dfrac{x}{2} - 12 = \dfrac{y}{4} + 8. \\[2ex] \dfrac{x+y}{5} - \dfrac{2y-x}{4} = 15. \end{cases}$

16. $\begin{cases} \dfrac{3x+7}{6} - \dfrac{7-2y}{10} = x. \\[2ex] \dfrac{2y-3}{6} - \dfrac{5-3x}{8} = y. \end{cases}$

14. $\begin{cases} x - \dfrac{3x+2}{5} = \dfrac{y+2}{3}. \\[2ex] y - \dfrac{2y+1}{3} = \dfrac{x-6}{5}. \end{cases}$

17. $\begin{cases} \dfrac{x+3y}{2x-y} = -\dfrac{3}{8}. \\[2ex] \dfrac{7y-x}{2+x+2y} = -17. \end{cases}$

18. $\begin{cases} \dfrac{x-5}{4} - \dfrac{2x-y-1}{3} = \dfrac{2y-2}{5}. \\[2ex] \dfrac{2y+x-1}{9} = \dfrac{x+y}{4}. \end{cases}$

19. $$\begin{cases} \dfrac{\dfrac{3x}{4}-\dfrac{y}{3}}{\dfrac{1}{2}}-\dfrac{\dfrac{x}{2}+\dfrac{2y}{5}}{\dfrac{13}{4}}=-\dfrac{7}{6}. \\ 4y-3x=11. \end{cases}$$

20. $$\begin{cases} \dfrac{3x-5y}{2}+3=\dfrac{2x+y}{5}. \\ 8-\dfrac{x-2y}{4}=\dfrac{x}{2}+\dfrac{y}{3}. \end{cases}$$

21. $$\begin{cases} x-\dfrac{2x+y}{3}=\dfrac{17}{12}-\dfrac{2y+x}{4}. \\ \dfrac{5}{4}-\dfrac{2x-y}{4}=y-\dfrac{2y-x}{3}. \end{cases}$$

22. $$\begin{cases} \dfrac{2x}{3}-\dfrac{3y}{5}-\dfrac{x+2y}{4}=3-\dfrac{5x-6y}{4}. \\ \dfrac{x}{2}+y-\dfrac{3x-y}{5}=-5+\dfrac{x}{15}. \end{cases}$$

23. $$\begin{cases} \dfrac{x-2y}{2x-4y-1}=\dfrac{3x}{6x-1}. \\ x-\dfrac{3-5y}{x+2}=\dfrac{4x-13}{4}. \end{cases}$$

24. $$\begin{cases} 4x^2+4xy+272=(x+y)(4x+17). \\ \dfrac{y(x-y)+54}{x-y}=\dfrac{5y+27}{5}. \end{cases}$$

25. $$\begin{cases} x^2-4y^2-17=(x+2y-2)(x-2y+1). \\ \dfrac{xy-5}{y-2}+\dfrac{1-2x}{y-1}=x. \end{cases}$$

Note. In solving literal simultaneous equations, the method of elimination by addition or subtraction is usually to be preferred.

26. Solve the equations $\begin{cases} ax + by = c & (1) \\ a'x + b'y = c' & (2) \end{cases}$

Multiplying (1) by b', $ab'x + bb'y = b'c$

Multiplying (2) by b, $a'bx + bb'y = bc'$

Subtracting, $ab'x - a'bx = b'c - bc'$

Whence, $x = \dfrac{b'c - bc'}{ab' - a'b}$

Multiplying (1) by a', $aa'x + a'by = a'c$ (3)

Multiplying (2) by a, $aa'x + ab'y = ac'$ (4)

Subtracting (3) from (4), $ab'y - a'by = ac' - a'c$

Whence, $y = \dfrac{ac' - a'c}{ab' - a'b}$

Solve the following equations:

27. $\begin{cases} 2x - 3y = a. \\ 3x + 4y = b. \end{cases}$

28. $\begin{cases} ax + by = m. \\ cx + dy = n. \end{cases}$

29. $\begin{cases} ax - by = c. \\ x - y = d. \end{cases}$

30. $\begin{cases} ax - by = 0. \\ mx + ny = p. \end{cases}$

31. $\begin{cases} ax + by = m. \\ cx - dy = n. \end{cases}$

32. $\begin{cases} mx - ny = p. \\ m'x - n'y = p'. \end{cases}$

33. $\begin{cases} \dfrac{x}{a} - \dfrac{y}{b} = m. \\ \dfrac{x}{c} + \dfrac{y}{d} = n. \end{cases}$

34. $\begin{cases} x + ay = a(a + 2b). \\ y - \dfrac{x}{b} = b. \end{cases}$

35. $\begin{cases} ax + by = 2. \\ ab(ay - bx) = a^2 - b^2. \end{cases}$

36. $\begin{cases} \dfrac{x}{a} + \dfrac{y}{b} = 2ab. \\ x + y = ab(a + b). \end{cases}$

37. $\begin{cases} mx + ny = \dfrac{m^4+n^4}{m^2n^2}. \\ nx + my = \dfrac{m^2+n^2}{mn}. \end{cases}$

38. $\begin{cases} (a+b)x-(a-b)y=4ab. \\ (a-b)x-(a+b)y=0. \end{cases}$

39. $\begin{cases} \dfrac{x+a}{b}+\dfrac{y+b}{a}=\dfrac{2(a^2+b^2)}{ab}. \\ \dfrac{x-b}{a}-\dfrac{y-a}{b}=\dfrac{a^2-b^2}{ab}. \end{cases}$

40. $\begin{cases} \dfrac{x}{a}+\dfrac{y}{m+n}=\dfrac{a^2+m^2-n^2}{a(m+n)}. \\ (m+n)^2(m-n)x=a^3y. \end{cases}$

41. $\begin{cases} \dfrac{x}{a+b}+\dfrac{y}{a-b}=\dfrac{1}{a^2-b^2}. \\ \dfrac{x}{a-b}+\dfrac{y}{a+b}=\dfrac{1}{a^2-b^2}. \end{cases}$

42. $\begin{cases} \dfrac{x}{a+b}+\dfrac{y}{a-b}=2a. \\ x-y=4ab. \end{cases}$

Note. Certain fractional equations, in which the unknown quantities occur in the denominators, are readily solved without previously clearing of fractions.

43. Solve the equations $\begin{cases} \dfrac{10}{x}-\dfrac{9}{y}=8 & (1) \\ \dfrac{8}{x}+\dfrac{15}{y}=-1 & (2) \end{cases}$

Multiplying (1) by 5, $\dfrac{50}{x}-\dfrac{45}{y}=40$

Multiplying (2) by 3, $\dfrac{24}{x}+\dfrac{45}{y}=-3$

Adding, $\dfrac{74}{x}=37$

$$37x = 74$$

$$x = 2$$

Substituting in (1), $$5 - \frac{9}{y} = 8$$

$$-\frac{9}{y} = 3$$

$$y = -3$$

Ans. $x = 2,\ y = -3.$

Solve the following equations:

44. $\begin{cases} \dfrac{3}{x} + \dfrac{1}{y} = \dfrac{5}{4}. \\ \dfrac{2}{x} - \dfrac{3}{y} = -1. \end{cases}$

45. $\begin{cases} \dfrac{2}{x} - \dfrac{3}{y} = -\dfrac{7}{5}. \\ \dfrac{15}{x} - \dfrac{8}{y} = -\dfrac{17}{3}. \end{cases}$

46. $\begin{cases} \dfrac{11}{x} - \dfrac{7}{y} = \dfrac{3}{2}. \\ \dfrac{2}{x} + \dfrac{4}{y} = -5. \end{cases}$

47. $\begin{cases} \dfrac{3}{x} - \dfrac{5}{2y} = 16. \\ \dfrac{1}{2x} + \dfrac{4}{y} = -15. \end{cases}$

48. $\begin{cases} \dfrac{m}{x} + \dfrac{n}{y} = 1. \\ \dfrac{n}{x} + \dfrac{m}{y} = 1. \end{cases}$

49. $\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = m. \\ \dfrac{c}{x} + \dfrac{d}{y} = n. \end{cases}$

50. $\begin{cases} \dfrac{2}{9x} - \dfrac{5}{2y} = -3. \\ \dfrac{5}{3x} + \dfrac{1}{4y} = \dfrac{17}{6}. \end{cases}$

51. $\begin{cases} \dfrac{m^2}{x} + \dfrac{n^2}{y} = mn(m+n). \\ \dfrac{n}{x} + \dfrac{m}{y} = m^2 + n^2. \end{cases}$

XV. SIMPLE EQUATIONS.

CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

188. If there are *three* simple equations containing *three* unknown quantities, we may combine two of them by the methods of elimination explained in the last chapter, so as to obtain an equation containing only two unknown quantities. We may then combine the third equation with either of the others, and obtain another equation containing the same two unknown quantities. By solving the equations thus obtained, we derive the values of two of the unknown quantities. These values being substituted in either of the given equations, the value of the third unknown quantity may be determined.

A similar method may be used when the number of equations and of unknown quantities is greater than three.

The method of elimination by addition or subtraction is usually the most convenient.

189. 1. Solve the equations

$$\begin{cases} 6x - 4y - 7z = 17 & (1) \\ 9x - 7y - 16z = 29 & (2) \\ 10x - 5y - 3z = 23 & (3) \end{cases}$$

Multiplying (1) by 3, $18x - 12y - 21z = 51$

Multiplying (2) by 2, $18x - 14y - 32z = 58$

Subtracting, $2y + 11z = -7$ (4)

Multiplying (1) by 5, $30x - 20y - 35z = 85$ (5)

Multiplying (3) by 3, $30x - 15y - 9z = 69$ (6)

Subtracting (5) from (6), $5y + 26z = -16$ (7)

$$\begin{array}{ll} \text{Multiplying (4) by 5,} & 10y+55z=-35 \\ \text{Multiplying (7) by 2,} & \underline{10y+52z=-32} \\ \text{Subtracting,} & 3z=-\ \ 3 \\ & z=-\ \ 1 \\ \text{Substituting in (4),} & 2y-11=-\ \ 7 \\ & \therefore y=\ \ \ 2 \end{array}$$

Substituting the values of y and z in (1),

$$6x-8+7=17$$

$$\therefore x=3$$

Ans. $x=3$, $y=2$, $z=-1$.

In certain cases the solution may be abridged by aid of the artifice which is employed in the following example.

2. Solve the equations $\begin{cases} u+x+y=6 & (1) \\ x+y+z=7 & (2) \\ y+z+u=8 & (3) \\ z+u+x=9 & (4) \end{cases}$

Adding the given equations,

$$3u+3x+3y+3z=30$$

$$\begin{array}{lll} \text{Whence,} & u+x+y+z=10 & (5) \\ \text{Subtracting (2) from (5),} & u=3 & \\ \text{Subtracting (3) from (5),} & x=2 & \\ \text{Subtracting (4) from (5),} & y=1 & \\ \text{Subtracting (1) from (5),} & z=4 & \end{array}$$

EXAMPLES.

Solve the following equations:

3. $\begin{cases} x+y=2. \\ y+z=-1. \\ z+x=3. \end{cases}$

4. $\begin{cases} 2x-5y=-19. \\ 3y+4z=13. \\ 2z-5x=12. \end{cases}$

5. $\begin{cases} 3x - 2y = -1. \\ 5y + 4z = -6. \\ x - y - 3z = 11. \end{cases}$

6. $\begin{cases} 2x - y = 5. \\ 3x + 2y - z = 6. \\ x - 3y + 2z = 1. \end{cases}$

7. $\begin{cases} x + y + z = 53. \\ x + 2y + 3z = 107. \\ x + 3y + 4z = 137. \end{cases}$

8. $\begin{cases} 3x - y - 2z = -23. \\ 6x + 2y + 3z = 15. \\ 4x + 3y - z = -6. \end{cases}$

9. $\begin{cases} x + y - z = 3. \\ y + z - x = 1. \\ z + x - y = -11. \end{cases}$

10. $\begin{cases} x - 2y + 3z = 0. \\ y - 2z + 3x = -25. \\ z - 2x + 3y = 9. \end{cases}$

11. $\begin{cases} 5x - 3y + 2z = 41. \\ 2x + y - z = 17. \\ 5x + 4y - 2z = 36. \end{cases}$

12. $\begin{cases} 2x + y + z = -2. \\ x + 2y + z = 0. \\ x + y + 2z = -4. \end{cases}$

13. $\begin{cases} 7x + 4y - z = -50. \\ 4x - 5y - 3z = 20. \\ x - 3y - 4z = 30. \end{cases}$

14. $\begin{cases} x - 6y + 4z = 3. \\ 4x + 4y - 3z = 10. \\ 2x + y + 6z = 46. \end{cases}$

15. $\begin{cases} 8x - 9y - 7z = -36. \\ 12x - y - 3z = 36. \\ 6x - 2y - z = 10. \end{cases}$

16. $\begin{cases} 4x - 3y + 2z = 40. \\ 5x + 9y - 7z = 47. \\ 9x + 8y - 3z = 97. \end{cases}$

17. $\begin{cases} \frac{x}{2} + \frac{y}{3} - \frac{z}{4} = -43. \\ \frac{x}{3} - \frac{y}{4} + \frac{z}{2} = 34. \\ \frac{x}{4} + \frac{y}{2} - \frac{z}{3} = -50. \end{cases}$

18. $\begin{cases} 2u - 3x = 1. \\ 3x - 4y = -1. \\ 4y - 5z = 1. \\ 5z - 6u = -2. \end{cases}$

19. $\begin{cases} 2y + z + 2u = -23. \\ y + 3z = -2. \\ 4x + z = 13. \\ \frac{x}{3} + 3u = -20. \end{cases}$

20. $\begin{cases} \frac{1}{x}+\frac{1}{y}=1. \\ \frac{1}{y}+\frac{1}{z}=\frac{3}{2}. \\ \frac{1}{z}+\frac{1}{x}=2. \end{cases}$

21. $\begin{cases} \frac{3}{x}-\frac{2}{y}=-13. \\ \frac{3}{y}+\frac{2}{z}=14. \\ \frac{3}{z}-\frac{2}{x}=18. \end{cases}$

22. $\begin{cases} ax+a^2y=2. \\ a^2y+a^3z=2. \\ a^3z+a^4x=a^3+1. \end{cases}$

23. $\begin{cases} 3u-z=22-x-2y. \\ 4x-y=35-3z. \\ 4u-2y=19-3x. \\ z=39-2u-4y. \end{cases}$

24. $\begin{cases} \frac{1}{x}+\frac{2}{y}+\frac{3}{z}=-7. \\ \frac{2}{x}-\frac{3}{y}+\frac{4}{z}=9. \\ \frac{3}{x}+\frac{1}{y}-\frac{2}{z}=5. \end{cases}$

25. $\begin{cases} y-z-\frac{x+z}{2}=1. \\ \frac{x-y}{5}-\frac{x-z}{6}=0. \\ \frac{y+z}{4}-\frac{x+y}{2}=-4. \end{cases}$

26. $\begin{cases} ay+bx=c. \\ cx+az=b. \\ bz+cy=a. \end{cases}$

27. $\begin{cases} 2-z-\frac{3x+y}{4}=0. \\ 8-\frac{y+16z}{3}=3x. \\ 25-12(x+z)=-y. \end{cases}$

28. $\begin{cases} \frac{2x+y}{4}-\frac{y-2z}{3}=1. \\ \frac{x+3y}{3}-\frac{x-z}{4}=-2. \\ \frac{z+y}{3}-\frac{z+x}{4}=-\frac{3}{2}. \end{cases}$

29.* $\begin{cases} ax+y-z=a^2+a-1. \\ ay+z-x=a^2-a+1. \\ az+x-y=a. \end{cases}$

30. $\begin{cases} x-ay+a^2z=a^3. \\ x-by+b^2z=b^3. \\ x-cy+c^2z=c^3. \end{cases}$

* Add the equations together.

XVI. PROBLEMS.

LEADING TO SIMPLE EQUATIONS CONTAINING MORE THAN ONE UNKNOWN QUANTITY.

190. In solving problems where more than one letter is used to represent the unknown quantities, we must obtain from the conditions of the problem *as many independent equations as there are unknown quantities to be determined.*

1. Divide 81 into two parts such that $\frac{3}{5}$ the greater shall exceed $\frac{5}{9}$ the less by 7.

Let $x =$ the greater part,

and $y =$ the less.

By the conditions, $\left\{\begin{aligned} x+y &= 81 \\ \frac{3x}{5} &= \frac{5y}{9}+7. \end{aligned}\right.$

Solving these equations, $x = 45$, $y = 36$.

2. If 3 be added to both numerator and denominator of a fraction, its value is $\frac{2}{3}$; and if 2 be subtracted from both numerator and denominator, its value is $\frac{1}{2}$. Required the fraction.

Let $x =$ the numerator,

and $y =$ the denominator.

By the conditions, $\left\{\begin{aligned} \frac{x+3}{y+3} &= \frac{2}{3} \\ \frac{x-2}{y-2} &= \frac{1}{2}. \end{aligned}\right.$

Solving these equations, $x = 7$, $y = 12$.

Therefore the fraction is $\frac{7}{12}$.

PROBLEMS.

3. Divide 50 into two parts such that three-eighths of the greater shall be equal to two-thirds of the less.

4. Find two numbers such that 7 times the greater exceeds $\frac{1}{7}$ the less by 97, and 7 times the less exceeds $\frac{1}{7}$ the greater by 47.

5. If one-fifth of A's age were added to two-thirds of B's, the sum would be $19\frac{1}{3}$ years; and if two-fifths of B's age were subtracted from seven-eighths of A's, the remainder would be $18\frac{1}{4}$ years. Required their ages.

6. If 1 be added to the numerator of a certain fraction, its value is $\frac{1}{3}$; and if 1 be added to its denominator, its value is $\frac{1}{4}$. Required the fraction.

7. A gentleman at the time of his marriage, found that his wife's age was $\frac{3}{4}$ of his own; but after they had been married 12 years, her age was $\frac{5}{6}$ of his. Required their ages at the time of their marriage.

8. A and B engaged in trade, A with $240 and B with $96. A lost twice as much as B; and on settling their accounts, it appeared that A had three times as much remaining as B. How much did each lose?

9. Eight years ago, A was 4 times as old as B; but in 12 years he will be only twice as old. Required their ages at present.

10. If 5 be added to both terms of a fraction, its value is $\frac{1}{2}$; and if 3 be subtracted from both, its value is $\frac{1}{4}$. Required the fraction.

11. A and B agreed to dig a well in 10 days; but having labored together 4 days, B agreed to finish the job, which he did in 16 days. In how many days could each of them alone dig the well?

12. If the greater of two numbers be divided by the less, the quotient is 2 and the remainder 12; but if 4 times the less be divided by the greater, the quotient is 1 and the remainder 14. Required the numbers.

13. If the numerator of a fraction be doubled, and the denominator increased by 7, its value is $\frac{2}{3}$; and if the denominator be doubled, and the numerator increased by 2, the value is $\frac{3}{5}$. Required the fraction.

14. If $a-1$ be subtracted from the numerator of a certain fraction, its value is $a+1$; and if a be added to its denominator, its value is a. Required the fraction.

15. A gentleman's two horses, with their harness, cost $300. The value of the poorer horse, with the harness, was $20 less than the value of the better horse; and the value of the better horse, with the harness, was twice that of the poorer horse. What was the value of each?

16. A merchant has three kinds of sugar. He sells 3 lbs. of the first quality, 4 lbs. of the second, and 2 lbs. of the third, for 60 cents; or, 4 lbs. of the first quality, 1 lb. of the second, and 5 lbs. of the third, for 59 cents; or, 1 lb. of the first quality, 10 lbs. of the second, and 3 lbs. of the third, for 90 cents. Required the price per pound of each quality.

17. A sum of money was divided equally between a certain number of persons. Had there been 3 more, each would have received $1 less; had there been 6 less, each would have received $5 more. How many persons were there, and how much did each receive?

Let $x =$ the number of persons,
and $y =$ what each received.
Then, $xy =$ the sum divided.

By the conditions,

$$\begin{cases} (x+3)(y-1) = xy \\ (x-6)(y+5) = xy. \end{cases}$$

Solving these equations, $x = 12$, $y = 5$.

18. A boy spent his money for oranges. If he had got five more for his money, they would have cost a half-cent each less; if three less, they would have cost a half-cent each more. How much money did he spend, and how many oranges did he get?

19. A merchant has two kinds of grain, worth 60 and 90 cents per bushel respectively. How many bushels of each kind must he take to make a mixture of 40 bushels, worth 80 cents per bushel?

20. My income and assessed taxes together amount to \$50. If the income tax were increased 50 per cent, and the assessed tax diminished 25 per cent, they would together amount to \$52.50. Required the amount of each tax.

21. A man purchased a certain number of eggs. If he had bought 20 more for the same money, they would have cost a cent apiece less; if 15 less, a cent apiece more. How many eggs did he buy, and at what price?

22. If a certain lot of land were 8 feet longer and 2 feet wider, it would contain 656 square feet more; and if it were 2 feet longer and 8 feet wider, it would contain 776 square feet more. Required its length and width.

23. If B gives A \$5, they will have equal amounts; but if A gives B \$15, B will have $\frac{7}{8}$ as much as A. How much money has each?

24. Find three numbers such that the first with half the other two, the second with one-third the other two, and the third with one-fourth the other two, may each be equal to 34.

25. There are four numbers whose sum is 136. Twice the first exceeds the second by 46, twice the second exceeds the third by 44, and twice the third exceeds the fourth by 40. Required the numbers.

26. The sum of the digits of a number of three figures is 13. If the number, decreased by 8, be divided by the sum of its second and third digits, the quotient is 25; and if 99 be added to the number, the digits will be inverted. Required the number.

Let $x =$ the first digit,
$y =$ the second,
and $z =$ the third.

Then, $100x + 10y + z =$ the number,
and $100z + 10y + x =$ the number with its digits inverted.

By the conditions,

$$x+y+z=13,$$

$$\frac{100x + 10y + z - 8}{y+z} = 25,$$

and $$100x + 10y + z + 99 = 100z + 10y + x.$$

Solving these equations, $x = 2$, $y = 8$, $z = 3$.

Therefore the number is 283.

27. The sum of the digits of a number of two figures is 11; and if 27 be subtracted from the number, the digits will be inverted. Required the number.

28. The sum of the digits of a number of three figures is 11, and the units' figure is twice the figure in the hundreds' place. If 297 be added to the number, the digits will be inverted. Required the number.

29. A and B can perform a piece of work in 6 days, A and C in 8 days, and B and C in 12 days. In how many days can each of them alone perform it?

30. If I were to make my field 5 rods longer and 4 rods wider, its area would be increased by 240 square rods; but if I were to make its length 4 rods less, and its width 5 rods less, its area would be diminished by 210 square rods. Required its length, width, and area.

31. Find three numbers such that the sum of the first and second is c, of the second and third is a, and of the third and first is b.

32. There is a number of three figures, whose digits have equal differences in their order. If the number be divided by half the sum of its digits, the quotient is 41; and if 396 be added to the number, the digits will be inverted. Required the number.

33. A sum of money is divided equally between a certain number of persons. Had there been m more, each would have received a dollars less; if n less, each would have received b dollars more. How many persons were there, and how much did each receive?

34. A gentleman left a sum of money to be divided between his four sons, so that the share of the eldest should be $\frac{1}{2}$ the sum of the shares of the other three, of the second $\frac{1}{3}$ the sum of the other three, and of the third $\frac{1}{4}$ the sum of the other three. It was found that the share of the eldest exceeded that of the youngest by $140. What was the whole sum, and how much did each receive?

35. A grocer bought a certain number of eggs, part at 2 for 5 cents and the rest at 3 for 8 cents, and paid for the whole $1.71. He sold them at 36 cents a dozen, and made 27 cents by the transaction. How many of each kind did he buy?

36. If a number of two figures be divided by the sum of its digits, the quotient is 7; and if the digits be inverted, the quotient of the resulting number, increased by 6, divided by the sum of the digits, is 5. Required the number.

37. If 45 be added to a certain number of two figures, the digits will be inverted; and if the resulting number be divided by the sum of its digits, the quotient is 7 and the remainder 6. Required the number.

38. A and B can do a piece of work in m days, B and C in n days, and C and A in p days. In what time can each alone perform the work?

39. A crew can row 10 miles in 50 minutes down stream, and 12 miles in an hour and a half against the stream. Find the rate in miles per hour of the current, and of the crew in still water.

Let $x =$ the rate of the crew in still water,
and $y =$ the rate of the current.

Then, $x + y =$ the rate rowing down stream,
and $x - y =$ the rate rowing up stream.

Since the distance divided by the rate gives the time, we have, by the conditions,

$$\begin{cases} \dfrac{10}{x+y} = \dfrac{5}{6} \\ \dfrac{12}{x-y} = \dfrac{3}{2}. \end{cases}$$

Solving these equations, $x = 10$, $y = 2$.

40. A crew can row a miles in b hours down stream, and c miles in d hours against the stream. Find the rate in miles per hour of the current, and of the crew in still water.

41. A boatman can row down stream a distance of 20 miles, and back again, in 10 hours; and he finds that he can row 2 miles against the current in the same time that he rows 3 miles with it. Required his time in going and in returning.

42. A number consists of three digits whose sum is 21. The sum of the first digit and twice the second exceeds the third by 8; and if 198 be added to the number, the digits will be inverted. Required the number.

43. A merchant has two casks of wine. He pours from the first cask into the second as much as the second contained at first; he then pours from the second into the first as much as was left in the first; and again from the first into the second as much as was left in the second. There are now 16 gallons in each cask. How many gallons did each contain at first?

44. A number consists of two figures. If the digits be inverted, the sum of the resulting number and the original number is 121; and if the number be divided by the sum of its digits, the quotient is 5 and the remainder 10. Required the number.

45. A man has \$30,000 invested at a certain rate of interest, and owes \$20,000, on which he pays interest at another rate; and the interest which he receives exceeds that which he pays by \$800. Another man has \$35,000 invested at the second rate of interest, and owes \$24,000, on which he pays interest at the first rate; and the interest which he receives exceeds that which he pays by \$310. What are the two rates of interest?

46. A certain sum of money, at simple interest, amounted in 2 years to \$132, and in 5 years to \$150. Required the sum, and the rate of interest.

47. A certain sum of money, at simple interest, amounted in m years to a dollars, and in n years to b dollars. Required the sum, and the rate of interest.

48. A train running from A to B meets with an accident which causes its speed to be reduced to one-third of what it was before, and it is in consequence 5 hours late. If the accident had happened 60 miles nearer B, the train would have been only 1 hour late. What was the rate of the train before the accident?

Let $3x =$ the rate of the train before the accident.
Then, $x =$ its rate after the accident.
Let $y =$ the distance to B from the point of detention.

By the conditions,
$$\frac{y}{x} = \frac{y}{3x} + 5$$
$$\frac{y-60}{x} = \frac{y-60}{3x} + 1.$$

Solving these equations, $x = 10$.

Hence the rate of the train before the accident was 30 miles an hour.

49. A man rows down a stream, whose rate is $3\frac{1}{2}$ miles per hour, for a certain distance in 1 hour and 40 minutes. In returning, it takes him 6 hours and 30 minutes to arrive at a point 2 miles short of his starting-place. Find the distance which he rowed down stream, and his rate of pulling.

50. If a certain number be divided by the sum of its two digits, the quotient is 6 and the remainder 1. If the digits be inverted, the quotient of the resulting number increased by 8, divided by the sum of the digits, is 6. Required the number.

51. A train running from A to B meets with an accident which delays it 30 minutes; after which it proceeds at three-fifths its former rate and arrives at B 2 hours and 30 minutes late. If the accident had occurred 30 miles nearer A, the train would have been 3 hours late. What was the rate of the train before the accident?

52. A, B, and C together have $24. A gives to B and C as much as each of them has; B gives to A and C as much as each of them then has; and C gives to A and B as much as each of them then has. They have now equal amounts. How much did each have at first?

53. A and B are building a fence 126 feet long. After 3 hours, A leaves off, and B finishes the work in 14 hours. If 7 hours had occurred before A left off, B would have finished the work in $4\frac{2}{3}$ hours. How many feet does each build in one hour?

54. Divide 115 into three parts such that the first part increased by 30, twice the second part, increased by 2, and 6 times the third part, increased by 4, may all be equal to each other.

55. Four men, A, B, C, and D, play at cards, B having $1 more than C. After A has won half of B's money, B one-third of C's, and C one-fourth of D's, A, B, and C have each $18. How much had each at first?

56. A gives to B and C as much as each of them has; B gives to A and C as much as each of them then has; and C gives to A and B as much as each of them then has. Each has now $48. How much did each have at first?

57. A, B, and C, were engaged to mow a field. The first day, A worked 2 hours, B 3 hours, and C 5 hours, and together they mowed 1 acre; the second day, A worked 4 hours, B 9 hours, and C 6 hours, and all together mowed 2 acres; the third day, A worked 10 hours, B 12 hours, and C 5 hours, and all together mowed 3 acres. In what time could each alone mow an acre?

58. A man invests $3600, partly in $3\frac{1}{2}$ per cent bonds, and partly in 4 per cent bonds. The income from the $3\frac{1}{2}$ per cent bonds exceeds the income from the 4 per cent bonds by $6. How much has he in each kind of bond?

59. A and B run a race of 480 feet. The first heat, A gives B a start of 48 feet, and beats him by 6 seconds; the second heat, A gives B a start of 144 feet, and is beaten by 2 seconds. How many feet can each run in a second?

60. The fore-wheel of a carriage makes 4 revolutions more than the hind-wheel in going 96 feet; but if the circumference of the fore-wheel were $\frac{3}{2}$ as great, and of the hind-wheel $\frac{4}{3}$ as great, the fore-wheel would make only 2 revolutions more than the hind-wheel in going the same distance. Find the circumference of each wheel.

61. A and B together can do a piece of work in $4\frac{4}{5}$ days; but if A had worked one-half as fast, and B twice as fast, they would have finished it in $4\frac{4}{11}$ days. In how many days could each alone perform the work?

62. A and B run a race of 300 yards. The first heat, A gives B a start of 40 yards, and beats him by 2 seconds; the second heat, A gives B a start of 16 seconds, and is beaten by 36 yards. How many yards can each run in a second?

XVII. INVOLUTION.

191. **Involution** is the process of raising a quantity to any required power.

This is effected, as is evident from Art. 13, by taking the quantity as a factor a number of times equal to the exponent of the required power.

192. If the quantity to be involved is positive, all its powers will evidently be positive; but if it is negative, all its *even* powers will be positive, and all its *odd* powers negative. Thus,

$$(-a)^2=(-a)\times(-a) \qquad\qquad\qquad\qquad =+a^2$$
$$(-a)^3=(-a)\times(-a)\times(-a) \qquad\qquad =-a^3$$
$$(-a)^4=(-a)\times(-a)\times(-a)\times(-a)=+a^4\text{; etc.}$$

Hence, *the* EVEN *powers of any quantity are positive; and the* ODD *powers of a quantity have the same sign as the quantity itself.*

INVOLUTION OF MONOMIALS.

193. 1. Find the value of $(5a^2c)^4$.

$$(5a^2c)^4=5a^2c\times 5a^2c\times 5a^2c\times 5a^2c=625a^8c^4,\ \textit{Ans.}$$

2. Find the value of $(-3m^4)^3$.

$$(-3m^4)^3=(-3m^4)\times(-3m^4)\times(-3m^4)=-27m^{12},\ \textit{Ans.}$$

From the above examples we derive the following rule:

Raise the numerical coefficient to the required power, and multiply the exponent of each letter by the exponent of the required power.

Give to every even power the positive sign, and to every odd power the sign of the quantity itself.

EXAMPLES.

Write by inspection the values of the following:

3. $(-ab^2c^3)^4$.	7. $(-b^2c^3)^5$.	11. $(3a^3b^4c)^6$.
4. $(-5a^2b)^3$.	8. $(a^2b^2c^m)^n$.	12. $(-6x^ry^{7s})^3$.
5. $(x^ny)^m$.	9. $(-5m^3n)^4$.	13. $(4a^mb^{2n})^5$.
6. $(2mn^2x^3)^6$.	10. $(4a^2b^3c^4)^3$.	14. $(-7x^6y^3z^{15})^3$.

A fraction is raised to any power by *raising both numerator and denominator to the required power.*

For example, $\left(-\dfrac{2x^m}{3y^2}\right)^4 = \dfrac{16x^{4m}}{81y^8}$.

Write by inspection the values of the following:

15. $\left(\dfrac{ac^2}{b^3}\right)^4$.	17. $\left(-\dfrac{4ax^2}{5b}\right)^2$.	19. $\left(-\dfrac{7xy^2}{3n}\right)^3$.
16. $\left(\dfrac{3a^2b^3}{4xy^4}\right)^3$.	18. $\left(\dfrac{2}{3}a^3x^2\right)^6$.	20. $\left(-\dfrac{bcx^n}{4a^2}\right)^5$.

SQUARE OF A POLYNOMIAL.

194. We find by multiplication:

$$
\begin{array}{llllll}
a+b+c \\
a+b+c \\
\hline
a^2 + ab + ac \\
\quad + ab & & + b^2 + bc \\
\quad\quad + ac & & \quad + bc + c^2 \\
\hline
a^2 + 2ab + 2ac + b^2 + 2bc + c^2
\end{array}
$$

This result, for convenience of enunciation, may be written as follows:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

In a similar manner, we find:

$$(a+b+c+d)^2 = a^2+b^2+c^2+d^2$$
$$+2ab+2ac+2ad+2bc+2bd+2cd;$$

and so on.

We have then the following rule for the square of any polynomial:

Write the square of each term, together with twice its product by each of the following terms.

EXAMPLES.

1. Square $2x^2-3x-5$.

The squares of the terms are $4x^4$, $9x^2$, and 25. Twice the first term into each of the following terms gives the results, $-12x^3$ and $-20x^2$; and twice the second term into the following term gives the result, $30x$. Hence,

$$(2x^2-3x-5)^2 = 4x^4+9x^2+25-12x^3-20x^2+30x$$
$$= 4x^4-12x^3-11x^2+30x+25, \textit{ Ans.}$$

Square the following expressions:

2. $a-b+c$.
3. $a+b-c$.
4. $2x^2+x+1$.
5. x^2-3x+1.
6. x^2+4x-2.
7. $2x^2-x-3$.
8. $3a^2-5a+4$.
9. $2x^2+5x-7$.
10. $x+2y-3z$.
11. x^3-2x+5.
12. $2x^3+3x^2+1$.
13. $3a^2-2ab-5b^2$.
14. $4m^2+mn^2-3n^4$.
15. $a-b-c+d$.
16. $a-b+c-d$.
17. $1+x+x^2+x^3$.
18. $3x^3-2x^2-x+4$.
19. x^3-4x^2-2x-3.

CUBE OF A BINOMIAL.

195. We find by multiplication:

$$\begin{array}{rl} (a+b)^2 = & a^2 + 2ab + b^2 \\ & a + b \\ \hline & a^3 + 2a^2b + ab^2 \\ & \qquad\quad a^2b + 2ab^2 + b^3 \\ \hline (a+b)^3 = & a^3 + 3a^2b + 3ab^2 + b^3 \end{array}$$

$$\begin{array}{rl} (a-b)^2 = & a^2 - 2ab + b^2 \\ & a - b \\ \hline & a^3 - 2a^2b + ab^2 \\ & \qquad - a^2b + 2ab^2 - b^3 \\ \hline (a-b)^3 = & a^3 - 3a^2b + 3ab^2 - b^3 \end{array}$$

That is,

The cube of the sum of two quantities is equal to the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

The cube of the difference of two quantities is equal to the cube of the first, minus three times the square of the first times the second, plus three times the first times the square of the second, minus the cube of the second.

EXAMPLES.

1. Find the cube of $a + 2b$.

$$(a+2b)^3 = a^3 + 3a^2(2b) + 3a(2b)^2 + (2b)^3$$
$$= a^3 + 6a^2b + 12ab^2 + 8b^3, \textit{ Ans.}$$

2. Find the cube of $2x - 3y^2$.

$$(2x-3y^2)^3 = (2x)^3 - 3(2x)^2(3y^2) + 3(2x)(3y^2)^2 - (3y^2)^3$$
$$= 8x^3 - 36x^2y^2 + 54xy^4 - 27y^6, \textit{ Ans.}$$

Find the cubes of the following:

3. $x+3$.	7. $3m^2-1$.	11. $2x^3-3x$.
4. $2x-1$.	8. x^2+4.	12. $6x^2+xy$.
5. $ab-cd$.	9. $a+5b$.	13. $3m+5n$.
6. $a+4b$.	10. $2x-5y$.	14. $3xy-4a^2$.

The cube of a trinomial may be found by the above method, if two of its terms be enclosed in a parenthesis and regarded as a single term.

15. Find the cube of x^2-2x-1.

$$\begin{aligned}(x^2-2x-1)^3 &= [(x^2-2x)-1]^3\\ &= (x^2-2x)^3-3(x^2-2x)^2+3(x^2-2x)-1\\ &= x^6-6x^5+12x^4-8x^3-3(x^4-4x^3+4x^2)\\ &\qquad +3(x^2-2x)-1\\ &= x^6-6x^5+9x^4+4x^3-9x^2-6x-1, \text{ } \textit{Ans.}\end{aligned}$$

Find the cubes of the following:

16. x^2-x-1.	18. $a+b-c$.	20. x^2+3x+1.
17. $a-b+1$.	19. x^2-2x+2.	21. $2x^2-3x-1$.

ANY POWER OF A BINOMIAL.

196. By actual multiplication, we obtain:

$$(a+b)^2=a^2+2ab+b^2$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4; \text{ etc.}$$

$$(a-b)^2=a^2-2ab+b^2$$

$$(a-b)^3=a^3-3a^2b+3ab^2-b^3$$

$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4; \text{ etc.}$$

In these results we observe the following laws:

I. The number of terms is one more than the exponent of the binomial.

II. The exponent of a in the first term is the same as the exponent of the binomial, and decreases by 1 in each succeeding term.

III. The exponent of b in the second term is 1, and increases by 1 in each succeeding term.

IV. The coefficient of the first term is 1; and of the second term, is the exponent of the binomial.

V. If the coefficient of any term be multiplied by the exponent of a in that term, and the result divided by the exponent of b increased by 1, the quotient will be the coefficient of the next term.

VI. If the second term of the binomial is negative, the terms in the result are alternately positive and negative.

By aid of the above laws, any power of a binomial may be written by inspection.

EXAMPLES.

1. Expand $(a+x)^5$.

The exponent of a in the first term is 5, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, 5; multiplying the coefficient of the second term by 4, the exponent of a in that term, and dividing the result by the exponent of x increased by 1, or 2, we have 10 for the coefficient of the third term; and so on. Hence,

$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5, \text{ Ans.}$$

Note. The coefficients of terms equally distant from the beginning and end of the expansion are equal. Thus the coefficients of the latter half of an expansion may be written out from the first half.

2. Expand $(1-x)^6$.

$$(1-x)^6 = 1^6 - 6\cdot 1^5\cdot x + 15\cdot 1^4\cdot x^2 - 20\cdot 1^3\cdot x^3 + 15\cdot 1^2\cdot x^4 - 6\cdot 1\cdot x^5 + x^6$$
$$= 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6, \textit{ Ans.}$$

Note. If the first term of the binomial is numerical, it is convenient to write the exponents at first without reduction. The result should afterwards be reduced to its simplest form.

Expand the following:

3. $(a-b)^5$. **7.** $(1-x)^4$. **11.** $(x-4)^4$.

4. $(a+b)^6$. **8.** $(x+y)^7$. **12.** $(a-3)^5$.

5. $(a-b)^7$. **9.** $(m-n)^6$. **13.** $(a+2)^5$.

6. $(x-1)^5$. **10.** $(2+x)^4$. **14.** $(x-2)^6$.

15. Expand $(3m-n^2)^4$.

$$(3m-n^2)^4 = [(3m)-(n^2)]^4$$
$$= (3m)^4 - 4(3m)^3(n^2) + 6(3m)^2(n^2)^2 - 4(3m)(n^2)^3 + (n^2)^4$$
$$= 81m^4 - 108m^3n^2 + 54m^2n^4 - 12mn^6 + n^8, \textit{ Ans.}$$

Note. If either term of the binomial has a coefficient or exponent other than unity, it should be enclosed in a parenthesis before applying the laws.

Expand the following:

16. $(a-3x)^5$. **18.** $(a^2+bc)^7$. **20.** $(2a^2+b)^6$.

17. $(3+2b)^4$. **19.** $(x^3-4)^4$. **21.** $(2m^3-3n^2)^4$.

XVIII. EVOLUTION.

197. If a quantity be resolved into any number of equal factors, one of these factors is called a **Root** of the quantity.

198. **Evolution** is the process of finding any required root of a quantity. This is effected, as is evident from the preceding article, by finding a quantity which, when raised to the proposed power, will produce the given quantity.

199. The **Radical Sign**, $\sqrt{\ }$, when prefixed to a quantity, indicates that some root of the quantity is to be found.

Thus, $\sqrt{a}$ indicates the *second* or *square* root of a;

$\sqrt[3]{a}$ indicates the *third* or *cube* root of a;

$\sqrt[4]{a}$ indicates the *fourth* root of a; and so on.

The *index* of the root is the figure written over the radical sign. When no index is written, the square root is understood.

EVOLUTION OF MONOMIALS.

200. Required the cube root of $a^3b^6c^9$.

By Art. 198, we are to find a quantity which, when raised to the third power, will produce $a^3b^6c^9$. That quantity is evidently ab^2c^3. Hence,

$$\sqrt[3]{a^3b^6c^9} = ab^2c^3.$$

That is, *any root of a monomial is obtained by dividing the exponent of each factor by the index of the required root.*

201. From the relation of a root to its corresponding power, it follows from Art. 192 that:

1. *The odd roots of a quantity have the same sign as the quantity itself.*

Thus, $\sqrt[3]{a^3} = a$, and $\sqrt[5]{-a^5} = -a$.

2. *The even roots of a positive quantity are either positive or negative.*

For the even powers of either a positive or a negative quantity are positive.

Thus, $\sqrt[4]{a^4} = a$ or $-a$; that is, $\sqrt[4]{a^4} = \pm a$.

Note. The sign $\pm$, called the *double sign*, is prefixed to a quantity when we wish to indicate that it is either $+$ or $-$.

3. *The even roots of a negative quantity are impossible.*

For no quantity when raised to an even power can produce a negative result. Such roots are called *imaginary* quantities.

202. From Arts. 200 and 201 we derive the following rule:

Extract the required root of the numerical coefficient, and divide the exponent of each letter by the index of the root.

Give to every even root of a positive quantity the sign $\pm$, and to every odd root of any quantity the sign of the quantity itself.

Note. Any root of a fraction may be found by taking the required root of each of its terms.

EXAMPLES.

1. Find the square root of $9a^4b^2c^6$.

By the rule, $\sqrt{9a^4b^2c^6} = \pm 3a^2bc^3$, *Ans.*

2. Find the fifth root of $-32a^{10}x^{5m}$.

$$\sqrt[5]{-32a^{10}x^{5m}} = -2a^2x^m, \text{ Ans.}$$

Find the values of the following:

3. $\sqrt[3]{-125x^3y^6}$.
4. $\sqrt{49a^4b^2c^{12}}$.
5. $\sqrt[5]{m^{15}n^5p^{10}}$.
6. $\sqrt[4]{16a^4b^8}$.
7. $\sqrt[3]{-8a^3b^6x^9}$.
8. $\sqrt{121a^{12}c^2}$.
9. $\sqrt[m]{a^{mn}b^{mp}}$.
10. $\sqrt{81a^{2n}x^{2m+2}}$.
11. $\sqrt[4]{81m^{16}n^{20}}$.
12. $\sqrt[5]{-243c^{5n}d^{10m}}$.
13. $\sqrt[6]{64a^{18}b^{24}c^6}$.
14. $\sqrt[3]{x^{3n+3}y^{9m-6}}$.

15. $\sqrt{\dfrac{9x^2y^4}{16m^6}}$.
17. $\sqrt[5]{-\dfrac{32x^{15}}{y^{10}}}$.
19. $\sqrt[3]{-\dfrac{64m^3n^6}{125}}$.

16. $\sqrt[3]{\dfrac{8a^3b^9}{27c^6}}$.
18. $\sqrt[4]{\dfrac{a^4}{81b^8c^4}}$.
20. $\sqrt[5]{\dfrac{a^{5m}}{243x^{10}}}$.

SQUARE ROOT OF POLYNOMIALS.

203. Since $(a+b)^2 = a^2 + 2ab + b^2$, we know that the square root of $a^2 + 2ab + b^2$ is $a + b$.

It is required to find a process by which, when the quantity $a^2 + 2ab + b^2$ is given, its square root, $a + b$, may be determined.

$$\begin{array}{r|l|l}
 & a^2 + 2ab + b^2 & a + b \\
 & a^2 & \\ \hline
2a + b & 2ab + b^2 & \\
 & 2ab + b^2 & \\ \hline
\end{array}$$

The square root of the first term is a, which is the first term of the root. Subtracting its square from the given expression, the remainder is $2ab + b^2$, or $(2a + b)b$. Dividing the first term of this remainder by $2a$, or twice the first term of the root, we obtain b, the second term. This being added to $2a$, gives the complete divisor $2a + b$; which, when multiplied by b, and the product, $2ab + b^2$, subtracted from the remainder, completes the operation.

From the above process we derive the following rule:

Arrange the terms according to the powers of some letter.

Find the square root of the first term, write it as the first term of the root, and subtract its square from the given expression.

Divide the first term of the remainder by twice the first term of the root, and add the quotient to the root and also to the divisor.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

EXAMPLES.

204. 1. Find the square root of $9x^4 - 30a^3x^2 + 25a^6$.

$$
\begin{array}{l|l|l}
 & 9x^4 - 30a^3x^2 + 25a^6 & 3x^2 - 5a^3, \textit{ Ans.} \\
 & 9x^4 & \\
\hline
6x^2 - 5a^3 & -30a^3x^2 + 25a^6 & \\
 & -30a^3x^2 + 25a^6 & \\
\hline
\end{array}
$$

The square root of the first term is $3x^2$, which is the first term of the root. Subtracting $9x^4$ from the given expression, we have $-30a^3x^2$ as the first term of the remainder. Dividing this by twice the first term of the root, $6x^2$, we obtain the second term of the root, $-5a^3$, which, added to $6x^2$, completes the divisor $6x^2 - 5a^3$. Multiplying this divisor by $-5a^3$, and subtracting the product from the remainder, there is no remainder. Hence, $3x^2 - 5a^3$ is the required square root.

2. Find the square root of

$$12x^5 - 14x^3 + 1 - 8x^4 + 9x^6 + 4x.$$

Arranging according to the descending powers of x,

$$
\begin{array}{l|l|l}
 & 9x^6 + 12x^5 - 8x^4 - 14x^3 + 4x + 1 & 3x^3 + 2x^2 - 2x - 1, \\
 & 9x^6 & \textit{Ans.} \\
\hline
6x^3 + 2x^2 & 12x^5 & \\
 & 12x^5 + 4x^4 & \\
\hline
6x^3 + 4x^2 - 2x & -12x^4 & \\
 & -12x^4 - 8x^3 + 4x^2 & \\
\hline
6x^3 + 4x^2 - 4x - 1 & -6x^3 - 4x^2 + 4x + 1 & \\
 & -6x^3 - 4x^2 + 4x + 1 & \\
\hline
\end{array}
$$

It will be observed that each trial-divisor is equal to the preceding complete divisor, with its last term doubled.

Note. Since every square root has the double sign (Art. 201), the result may be written in a different form by changing the sign of each term. Thus, in Example 2, another form of the answer is

$$-3x^3 - 2x^2 + 2x + 1.$$

Find the square roots of the following:

3. $a^4-4a^3+6a^2-4a+1.$

4. $4x^4-4x^3-3x^2+2x+1.$

5. $9-12x+10x^2-4x^3+x^4.$

6. $19x^2+6x^3+25+x^4+30x.$

7. $40x+25-14x^2+9x^4-24x^3.$

8. $m^2+2m-1-\frac{2}{m}+\frac{1}{m^2}.$

9. $4a^4+64b^4-20a^3b-80ab^3+57a^2b^2.$

10. $28x^3+4x^4-14x+1+45x^2.$

11. $a^2+b^2+c^2-2ab-2ac+2bc.$

12. $x^2+4y^2+9z^2-4xy+6xz-12yz.$

13. $9x^6+30x^5+25x^4-42x^3-70x^2+49.$

14. $16c^6-40c^4-24c^3+25c^2+30c+9.$

15. $9+a^6+30a-4a^5+13a^2+14a^4-14a^3.$

16. $4x^6-4x^5y-3x^4y^2-6x^3y^3+5x^2y^4+4xy^5+4y^6.$

17. $25x^4-44x^3-40x+4x^6+25+46x^2-12x^5.$

18. $\frac{a^4}{9}-\frac{2a^3b}{3}+\frac{4a^2b^2}{3}-ab^3+\frac{b^4}{4}.$

19. $9x^6-12x^5y+10x^4y^2-16x^3y^3+9x^2y^4-4xy^5+4y^6.$

Find to four terms the approximate square roots of the following:

20. $1+x.$

21. $1-2a.$

22. $a^2-4ab+b^2.$

23. $4x^2+2y.$

SQUARE ROOT OF NUMBERS.

205. The method of Art. 204 may be used to extract the square roots of arithmetical numbers.

The square root of 100 is 10; of 10,000 is 100; etc. Hence, the square root of a number less than 100 is less than 10; the square root of a number between 10,000 and 100 is between 100 and 10; and so on.

That is, the integral part of the square root of a number of one or two figures, contains *one* figure; of a number of three or four figures, contains *two* figures; and so on. Hence,

If a point be placed over every second figure in any integral number, beginning with the units' place, the number of points shows the number of figures in the integral part of its square root.

206. Let it be required to find the square root of 4624.

$$\begin{array}{r|r|l}
\multicolumn{2}{r|}{4\dot{6}2\dot{4}} & 60+8 \\
\multicolumn{2}{r|}{a^2=3600} & =a+b \\
\hline
120+8 & 1024 & \\
=2a+b & 1024 & \\
\cline{2-2}
\end{array}$$

Pointing the number according to the rule of Art. 205, we see that there are two figures in the integral part of the square root.

Let a denote the value of the number in the tens' place in the root, and b the number in the units' place. Then a must be the greatest multiple of 10 whose square is less than 4624; this we find to be 60. Subtracting a^2, that is, the square of 60 or 3600, from the given number, the remainder is 1024. Dividing the remainder by $2a$ or 120, we have 8 as the value of b. Adding this to 120, multiplying the result by 8, and subtracting the product, 1024, there is no remainder. Hence, $60+8$ or 68 is the required square root.

The ciphers being omitted for the sake of brevity, the work will stand as follows:

$$\begin{array}{r|r|l}
\multicolumn{2}{r|}{4\dot{6}2\dot{4}} & 68 \\
\multicolumn{2}{r|}{36} & \\
\hline
128 & 1024 & \\
 & 1024 & \\
\cline{2-2}
\end{array}$$

From the above process we derive the following rule:

Separate the number into periods by pointing every second figure, beginning with the units' place.

Find the greatest square in the left-hand period, and write its square root as the first figure of the root; subtract its square from the number, and to the result bring down the next period.

Divide this remainder, omitting the last figure, by twice the part of the root already found, and annex the quotient to the root and also to the divisor.

Multiply the complete divisor by the figure of the root last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, doubling the part of the root already found for the next trial-divisor.

Note 1. It should be observed that decimals require to be pointed to the *right*.

Note 2. As the trial-divisor is an *incomplete* divisor, it is sometimes found that after completion it gives a product greater than the remainder. In such a case, the last root-figure is too large, and one less must be substituted for it.

Note 3. If any root-figure is 0, annex 0 to the trial-divisor, and bring down to the remainder the next period.

EXAMPLES.

207. **1.** Find the square root of 49.449024.

```
          49.449024 | 7.032, Ans.
          49        |
        ------------
 1403 | 4490
      | 4209
       --------
14062 |  28124
      |  28124
```

Since the second root-figure is 0, we annex 0 to the trial-divisor 14, and bring down to the remainder the next period, 90.

Extract the square roots of the following:

2. 45796.
3. 273529.
4. 654481.
5. 33.1776.
6. .247009.
7. .081796.
8. .521284.
9. 1.170724.
10. 446.0544.
11. .0022448644.
12. 811440.64.
13. .68112009.

If there is a final remainder, the given number has no exact square root; but we may continue the operation by annexing periods of ciphers, and thus obtain an approximate value of the square root, correct to any desired number of decimal places.

14. Extract the square root of 12 to five figures.

```
        12.00000000 | 3.4641..., Ans.
         9          |
      -------------
   64 | 300
      | 256
   -----------
  686 | 4400
      | 4116
  -----------
 6924 | 28400
      | 27696
 ------------
69281 | 70400
      | 69281
 ------------
         1119
```

Extract the square roots of the following to five figures:

15. 2.
16. 3.
17. 5.
18. 11.
19. 31.
20. 17.3.
21. .7.
22. .08.
23. .144.
24. .001.
25. .00625.
26. 2.08627.

The square root of a fraction may be obtained by taking the square roots of its terms.

If the denominator is not a perfect square, it is better to reduce the fraction to an equivalent fraction whose denominator is a perfect square.

Thus, to obtain the square root of $\frac{3}{8}$, we should proceed as follows:

$$\sqrt{\frac{3}{8}} = \sqrt{\frac{6}{16}} = \frac{\sqrt{6}}{4} = \frac{2.44949\ldots}{4} = .61237\ldots$$

Extract the square roots of the following to five figures:

27. $\frac{7}{4}$. 29. $\frac{10}{9}$. 31. $\frac{4}{3}$. 33. $\frac{11}{8}$. 35. $\frac{7}{18}$.

28. $\frac{3}{16}$. 30. $\frac{1}{5}$. 32. $\frac{5}{12}$. 34. $\frac{9}{10}$. 36. $\frac{13}{72}$.

CUBE ROOT OF POLYNOMIALS.

208. Since $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we know that the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$.

It is required to find a process by which, when the quantity $a^3 + 3a^2b + 3ab^2 + b^3$ is given, its cube root, $a + b$, may be determined.

$$\begin{array}{r|l|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 & a+b \\
 & a^3 & \\
\hline
3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 & \\
 & 3a^2b + 3ab^2 + b^3 & \\
\hline
\end{array}$$

The cube root of the first term is a, which is the first term of the root. Subtracting its cube from the given expression, the remainder is $3a^2b + 3ab^2 + b^3$, or $(3a^2 + 3ab + b^2)b$. Dividing the first term of this remainder by $3a^2$, or three times the square of the first term of the root, we obtain b, the second term.

Adding to the trial-divisor $3ab$, that is, three times the product of the first term of the root by the second, and b^2, that is, the square of the last term of the root, completes the divisor, $3a^2 + 3ab + b^2$. This being multiplied by b, and the product, $3a^2b + 3ab^2 + b^3$, subtracted from the remainder, completes the operation.

From the above process we derive the following rule:

Arrange the terms according to the powers of some letter.

Find the cube root of the first term, write it as the first term of the root, and subtract its cube from the given expression.

Divide the first term of the remainder by three times the square of the first term of the root, and write the quotient as the next term of the root.

Add to the trial-divisor three times the product of the first term of the root by the second, and the square of the second term.

Multiply the complete divisor by the term of the root last obtained, and subtract the product from the remainder.

If other terms remain, proceed as before, taking three times the square of the root already found for the next trial-divisor.

EXAMPLES.

209. 1. Find the cube root of $8x^6 - 36x^4y + 54x^2y^2 - 27y^3$.

$$\begin{array}{r|l|l}
 & 8x^6 - 36x^4y + 54x^2y^2 - 27y^3 & 2x^2 - 3y, \textit{ Ans.} \\
 & 8x^6 & \\
\hline
12x^4 - 18x^2y + 9y^2 & -36x^4y + 54x^2y^2 - 27y^3 & \\
 & -36x^4y + 54x^2y^2 - 27y^3 & \\
\hline
\end{array}$$

The cube root of the first term is $2x^2$, which is the first term of the root. Subtracting $8x^6$ from the given expression, we have $-36x^4y$ as the first term of the remainder. Dividing this by three times the square of the first term of the root, $12x^4$, we obtain $-3y$ as the second term of the root. Adding to the trial-divisor three times the product of the first term of the root by the second, $-18x^2y$, and the square of the second term, $9y^2$, completes the divisor, $12x^4 - 18x^2y + 9y^2$. Multiplying this by $-3y$, and subtracting the product from the remainder, there is no remainder. Hence, $2x^2 - 3y$ is the required cube root.

2. Find the cube root of $40x^3 - 6x^5 - 96x + x^6 - 64$.

Arranging according to the descending powers of x,

$$\begin{array}{l|l|l}
 & x^6-6x^5+40x^3-96x-64 & x^2-2x-4, \\
 & x^6 & \qquad\qquad \textit{Ans.} \\
\hline
3x^4-6x^3+4x^2 & -6x^5 & \\
 & -6x^5+12x^4-8x^3 & \\
\hline
3x^4-12x^3+12x^2 & -12x^4+48x^3-96x-64 & \\
\qquad -12x^2+24x+16 & & \\
\hline
3x^4-12x^3 \qquad +24x+16 & -12x^4+48x^3-96x-64 & \\
\hline
\end{array}$$

The second complete divisor is formed as follows:

The trial-divisor is 3 times the square of the root already found; that is, $3(x^2-2x)^2$, or $3x^4-12x^3+12x^2$. Three times the product of the root already found by the last term of the root is $3(-4)(x^2-2x)$, or $-12x^2+24x$; and the square of the last root-term is 16. Adding these, we have for the complete divisor $3x^4-12x^3+24x+16$.

Find the cube roots of the following:

3. $1-6y+12y^2-8y^3$.

4. $27x^6+27x^4+9x^2+1$.

5. $54xy^2+27y^3+36x^2y+8x^3$.

6. $64a^3-144a^2xy+108ax^2y^2-27x^3y^3$.

7. $x^6+6x^5-40x^3+96x-64$.

8. $y^6-1+5y^3-3y^5-3y$.

9. $15x^4-6x-6x^5+15x^2+1+x^6-20x^3$.

10. $9x^3-21x^2-36x^5+8x^6-9x+42x^4-1$.

11. $8a^6-12a^5-54a^4+59a^3+135a^2-75a-125$.

12. $30x^2-12x^5-12x+8-25x^3+8x^6+30x^4$.

13. $x^6+3x^5y-3x^4y^2-11x^3y^3+6x^2y^4+12xy^5-8y^6$.

14. $27a^6-54a^5b+9a^4b^2+28a^3b^3-3a^2b^4-6ab^5-b^6$.

CUBE ROOT OF NUMBERS.

210. The method of Art. 209 may be used to extract the cube roots of arithmetical numbers.

The cube root of 1000 is 10; of 1,000,000 is 100; etc. Hence, the cube root of a number less than 1000 is less than 10; the cube root of a number between 1,000,000 and 1000 is between 100 and 10; and so on.

That is, the integral part of the cube root of a number of one, two, or three figures, contains *one* figure; of a number of four, five, or six figures, contains *two* figures; and so on. Hence,

If a point be placed over every third figure in any integral number, beginning with the units' place, the number of points shows the number of figures in the integral part of its cube root.

211. Let it be required to find the cube root of 157464.

$$
\begin{array}{rr|r|l}
 & & 15\dot{7}46\dot{4} & 50+4 \\
 & a^3 = & 125000 & =a+b \\
\hline
3a^2 = & 7500 & 32464 & \\
3ab = & 600 & & \\
b^2 = & 16 & & \\
\cline{2-2}
 & 8116 & 32464 & \\
\cline{3-3}
\end{array}
$$

Pointing the number according to the rule of Art. 210, we see that there are two figures in the integral part of the cube root.

Let a denote the value of the number in the tens' place in the root, and b the number in the units' place. Then a must be the greatest multiple of 10 whose cube is less than 157464; this we find to be 50. Subtracting a^3, that is, the cube of 50 or 125000, from the given number, the remainder is 32464. Dividing this remainder by $3a^2$, that is, 3 times the square of 50 or 7500, we obtain 4 as the value of b. Adding to the trial-divisor $3ab$, that is, 3 times the product of 50 and 4, or 600, and b^2, or 16, we have the complete divisor 8116. Multiplying this by 4, and subtracting the product, 32464, there is no remainder. Hence, $50+4$ or 54 is the required cube root.

The ciphers being omitted for the sake of brevity, the work will stand as follows:

$$\begin{array}{r|l|l}
 & 15\dot{7}46\dot{4} & 54 \\
 & 125 & \\ \hline
7500 & 32464 & \\
600 & & \\
\underline{16} & & \\
8116 & 32464 & \\ \cline{2-2}
\end{array}$$

From the above process, we derive the following rule:

Separate the number into periods by pointing every third figure, beginning with the units' place.

Find the greatest cube in the left-hand period, and write its cube root as the first figure of the root; subtract its cube from the number, and to the result bring down the next period.

Divide this remainder by three times the square of the root already found, with two ciphers annexed, and write the quotient as the next figure of the root.

Add to the trial-divisor three times the product of the last root-figure and the part of the root previously found, with one cipher annexed, and the square of the last root-figure.

Multiply the complete divisor by the figure of the root last obtained, and subtract the product from the remainder.

If other periods remain, proceed as before, taking three times the square of the root already found for the next trial-divisor.

The notes to Art. 206 apply with equal force to examples in cube root, except that in Note 3 *two* ciphers should be annexed to the trial-divisor.

212. In the illustration of Art. 208, if there had been more terms in the given quantity, the next trial-divisor would have been three times the square of $a+b$; that is, $3a^2+6ab+3b^2$. We observe that this is obtained from the preceding complete divisor, $3a^2+3ab+b^2$, by adding to it its second term, $3ab$, and twice its third term, $2b^2$. We may

then use the following rule for forming the successive trial-divisors in the cube root of numbers:

To the preceding complete divisor, add its second term and twice its third term; and annex two ciphers to the result.

EXAMPLES.

213. 1. Find the cube root of 8.144865728.

	8.144865728	2.012, *Ans.*
	8	
120000	144865	
600		
1		
120601	120601	
600	24264728	
2		
12120300		
12060		
4		
12132364	24264728	

Since the second root-figure is 0, we annex two ciphers to the trial-divisor 1200, and bring down to the remainder the next period, 865.

The second trial-divisor is formed by the rule of Art. 212. The preceding complete divisor is 120601; adding its second term, 600, and twice its third term, 2, we have 121203; annexing two ciphers to this, we obtain the result 12120300.

Extract the cube roots of the following:

2. 29791.
3. 97.336.
4. .681472.
5. 1860867.
6. 1.481544.
7. .000941192.
8. 8.242408.
9. 51478848.
10. 10077.696.
11. .517781627.
12. 116.930169.
13. .031855013.
14. .724150792.
15. 1039509.197.
16. .000152273304.

Extract the cube roots of the following to four figures:

17. 2. **19.** 7.2. **21.** $\frac{3}{8}$. **23.** $\frac{7}{27}$.

18. 6. **20.** .03. **22.** $\frac{5}{4}$. **24.** $\frac{2}{3}$.

214. When the index of the required root is the product of two or more numbers, we may obtain the result by *successive extractions of the simpler roots.*

For, by Art. 198, $(\sqrt[mn]{a})^{mn} = a.$

Taking the nth root of both members,

$$(\sqrt[mn]{a})^m = \sqrt[n]{a}. \qquad (1)$$

Taking the mth root of both members of (1),

$$\sqrt[mn]{a} = \sqrt[m]{(\sqrt[n]{a})}.$$

That is,

The mnth root of a quantity is equal to the mth root of the nth root of that quantity.

For example, the fourth root is the square root of the square root; the sixth root is the cube root of the square root; etc.

EXAMPLES.

Find the fourth roots of the following:

1. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4.$
2. $x^8 - 4x^7 + 10x^6 - 16x^5 + 19x^4 - 16x^3 + 10x^2 - 4x + 1.$
3. $x^8 - 8x^7 + 16x^6 + 16x^5 - 56x^4 - 32x^3 + 64x^2 + 64x + 16.$

Find the sixth roots of the following:

4. $a^{12} - 6a^{10} + 15a^8 - 20a^6 + 15a^4 - 6a^2 + 1.$
5. $64x^6 + 192x^5 + 240x^4 + 160x^3 + 60x^2 + 12x + 1.$

XIX. THE THEORY OF EXPONENTS.

215. In the preceding chapters we have considered an exponent only as a positive whole number. It is, however, found convenient to employ fractional and negative exponents; and we proceed to define them, and to prove the rules for their use.

216. In Art. 13 we defined a positive integral exponent as indicating how many times a quantity was taken as a factor; thus,

a^m signifies $a \times a \times a \times \cdots$ to m factors.

We have also found the following rules to hold when m and n are positive integers:

$$\text{I.} \quad a^m \times a^n = a^{m+n}. \qquad \text{(Art. 79.)}$$

$$\text{II.} \quad (a^m)^n = a^{mn}. \qquad \text{(Art. 193.)}$$

217. The definition of Art. 13 has no meaning unless the exponent is a positive integer, and we must therefore adopt new definitions for fractional and negative exponents. It is convenient to have all forms of exponents subject to the same laws in regard to multiplication, division, etc., and we shall therefore assume Rule I. to hold for *all* values of m and n, and find what meanings must be attached to fractional and negative exponents in consequence.

218. Required the meaning of $a^{\frac{5}{3}}$.

Since Rule I. is to hold universally, we must have

$$a^{\frac{5}{3}} \times a^{\frac{5}{3}} \times a^{\frac{5}{3}} = a^{\frac{5}{3}+\frac{5}{3}+\frac{5}{3}} = a^5.$$

That is, $a^{\frac{5}{3}}$ is such a quantity that when raised to the third power the result is a^5. Hence (Art. 198), $a^{\frac{5}{3}}$ must be the cube root of a^5; or, $a^{\frac{5}{3}} = \sqrt[3]{a^5}$.

We will now consider the general case:

Required the meaning of $a^{\frac{p}{q}}$, *where* p *and* q *are positive integers.*

By Rule I., $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots$ to q factors

$$= a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}+\cdots \text{to } q \text{ terms}} = a^{\frac{p}{q}\times q} = a^p.$$

That is, $a^{\frac{p}{q}}$ is such a quantity that when raised to the qth power the result is a^p. Therefore $a^{\frac{p}{q}}$ must be the qth root of a^p; or,

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Hence, in a fractional exponent, *the numerator denotes a power and the denominator a root.*

For example, $a^{\frac{3}{4}} = \sqrt[4]{a^3}$; $c^{\frac{5}{2}} = \sqrt{c^5}$; $x^{\frac{1}{3}} = \sqrt[3]{x}$; etc.

EXAMPLES.

219. Express the following with radical signs:

1. $a^{\frac{1}{4}}$. 3. $2c^{\frac{1}{2}}$. 5. $x^{\frac{3}{4}}y^{\frac{2}{3}}$. 7. $4a^{\frac{m}{5}}b^{\frac{n}{6}}$. 9. $5y^{\frac{4}{7}}z^{\frac{9}{2}}$.

2. $b^{\frac{3}{7}}$. 4. $3am^{\frac{5}{4}}$. 6. $m^{\frac{3}{5}}n^{\frac{5}{3}}$. 8. $2c^{\frac{3}{8}}d^{\frac{2}{5}}$. 10. $ab^{\frac{1}{3}}c^{\frac{4}{5}}d^{\frac{7}{2}}$.

Express the following with fractional exponents:

11. $\sqrt[5]{x^6}$. 13. $\sqrt{n}$. 15. $3\sqrt{m^5}$. 17. $\sqrt[3]{a^4}\sqrt[5]{b}$.

12. $\sqrt[3]{y^2}$. 14. $\sqrt[3]{c}$. 16. $4\sqrt[7]{a^9}$. 18. $\sqrt{x^5}\sqrt[5]{y^2}$.

19. $5\sqrt{m^r}\sqrt[3]{n^s}$. 20. $2a\sqrt[n]{x}\sqrt[n]{y^m}$.

The value of a numerical quantity affected with a fractional exponent may be found by first extracting the root indicated by the denominator, and then raising the result to the power indicated by the numerator.

Thus, $(-8)^{\frac{2}{3}} = (\sqrt[3]{-8})^2 = (-2)^2 = 4.$

Find the values of the following:

21. $9^{\frac{5}{2}}$.	**23.** $36^{\frac{3}{2}}$.	**25.** $(-27)^{\frac{5}{3}}$.	**27.** $64^{\frac{7}{6}}$.
22. $27^{\frac{4}{3}}$.	**24.** $16^{\frac{5}{4}}$.	**26.** $(-32)^{\frac{4}{5}}$.	**28.** $(-216)^{\frac{4}{3}}$.

220. *Required the meaning of* a^0.

Since Rule I. is to hold universally, we must have

$$a^m \times a^0 = a^{m+0} = a^m.$$

Therefore a^0 must be equal to 1.

That is, *any quantity whose exponent is* 0 *is equal to* 1.

221. We pass next to the case of negative exponents.

Required the meaning of a^{-3}.

By Rule I., $a^{-3} \times a^3 = a^{-3+3} = a^0 = 1.$ (Art. 220.)

Hence, $a^{-3} = \dfrac{1}{a^3}.$

We will now consider the general case:

Required the meaning of a^{-n}, *n being integral or fractional.*

By Rule I., $a^{-n} \times a^n = a^{-n+n} = a^0 = 1.$

Hence, $a^{-n} = \dfrac{1}{a^n}.$

For example,

$$a^{-2} = \frac{1}{a^2};\ a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}};\ 3x^{-1}y^{-\frac{1}{2}} = \frac{3}{xy^{\frac{1}{2}}};\ \text{etc.}$$

222. In connection with Art. 221, the following principle may be noticed:

Any factor of the numerator of a fraction may be transferred to the denominator, or any factor of the denominator to the numerator, if the sign of its exponent be changed.

Thus, the fraction $\frac{a^2b^3}{cd^4}$ can be written in any of the forms

$$\frac{b^3}{a^{-2}cd^4},\quad \frac{a^2b^3c^{-1}}{d^4},\quad \frac{1}{a^{-2}b^{-3}cd^4},\ \text{etc.}$$

EXAMPLES.

223. Write the following with positive exponents:

1. x^3y^{-5}.
2. $x^{-1}y^{\frac{2}{3}}$.
3. $m^2n^{-\frac{1}{2}}$.
4. $4xy^{-\frac{1}{3}}$.
5. $a^{-1}b^{-2}$.
6. $3a^{\frac{1}{2}}b^{-\frac{2}{5}}$.
7. $2x^{-4}y^{-\frac{3}{7}}$.
8. $a^{-5}b^{-2}c^3$.
9. $5a^{-3}b^{-2}c$.
10. $2m^{-6}n^{-4}$.
11. $3x^{-\frac{2}{3}}y^{-\frac{2}{7}}$.
12. $a^{-2}b^{-\frac{3}{4}}c^{-\frac{4}{5}}$.

Transfer the literal factors from the denominators to the numerators in the following:

13. $\frac{1}{x}$.
14. $\frac{a^2}{x^3}$.
15. $\frac{3}{x^{-2}}$.
16. $\frac{1}{2x^{\frac{3}{4}}}$.
17. $\frac{3c}{x^2y^{-1}}$.
18. $\frac{ab^2}{cd^{\frac{1}{2}}}$.
19. $\frac{5a^2}{2bc^3}$.
20. $\frac{a^3}{2x^{\frac{2}{3}}y^{\frac{3}{5}}}$.
21. $\frac{3x}{5m^{-4}n^{-\frac{2}{5}}}$.

Transfer the literal factors from the numerators to the denominators in the following:

22. $\frac{2x^2}{3}$.
23. $\frac{3x^{\frac{1}{2}}}{4a}$.
24. $\frac{x^{-3}}{2}$.
25. $\frac{2c^{-\frac{3}{4}}}{5}$.
26. $3a^{\frac{5}{4}}$.
27. $\frac{5a^{-2}c}{b^{\frac{1}{2}}}$.
28. $m^{-\frac{2}{3}}n^{\frac{1}{2}}$.
29. $\frac{x^{-1}y^{\frac{3}{5}}}{z^2}$.
30. $\frac{4a^{-2}b^{-\frac{3}{2}}}{3c^3}$.

224. Since the definitions of fractional and negative exponents were obtained on the supposition that Rule I., Art. 216, was to hold universally, we have for any values of m and n,

$$a^m \times a^n \;\; = a^{m+n}.$$

For example,

$$a^2 \times a^{-5} = a^{2-5} \;\; = a^{-3};$$

$$a^{\frac{3}{4}} \times a^{-\frac{2}{3}} = a^{\frac{3}{4}-\frac{2}{3}} = a^{\frac{1}{12}};$$

$$a \;\times a^{-\frac{5}{2}} = a^{1-\frac{5}{2}} = a^{-\frac{3}{2}};\text{ etc.}$$

EXAMPLES.

Find the values of the following:

1. $a^3 \times a^{-1}$.
2. $a^2 \times a^{-2}$.
3. $x^{-1} \times x^{-5}$.
4. $n^{\frac{3}{4}} \times n^{-\frac{1}{3}}$.
5. $2x^{\frac{5}{2}} \times x^{-\frac{3}{2}}$.
6. $3a \times a^{-\frac{2}{3}}$.
7. $5c^{-3} \times 3c^{-\frac{1}{2}}$.
8. $a^3 \times \sqrt[3]{a^2}$.
9. $x^{-1} \times \sqrt[4]{x^{-3}}$.
10. $m^2 \times \dfrac{4}{\sqrt[5]{m}}$.
11. $2c^{-\frac{2}{7}} \times 3a\sqrt[5]{c^3}$.
12. $2a^{-3}b^{\frac{3}{4}} \times ab^{-1}$.
13. $x^2y^{-\frac{5}{3}} \times \dfrac{x^{-2}y^{\frac{8}{3}}}{2}$.
14. $\sqrt[6]{x} \times 5\sqrt{x^{-5}}$.
15. $\dfrac{1}{a^{\frac{1}{2}}b^{-2}} \times \dfrac{3}{a^{-3}b^{\frac{1}{3}}}$.

16. Multiply $a + 2a^{\frac{2}{3}} - 3a^{\frac{1}{3}}$ by $2 - 4a^{-\frac{1}{3}} - 6a^{-\frac{2}{3}}$.

$$\begin{array}{llll}
a + 2a^{\frac{2}{3}} & - 3a^{\frac{1}{3}} & & \\
2 - 4a^{-\frac{1}{3}} & - 6a^{-\frac{2}{3}} & & \\
\hline
2a + 4a^{\frac{2}{3}} & - 6a^{\frac{1}{3}} & & \\
\quad\;\; - 4a^{\frac{2}{3}} & - 8a^{\frac{1}{3}} & + 12 & \\
 & - 6a^{\frac{1}{3}} & - 12 & + 18a^{-\frac{1}{3}} \\
\hline
2a & - 20a^{\frac{1}{3}} & & + 18a^{-\frac{1}{3}}, \textit{ Ans.}
\end{array}$$

Note. In examples like the above, it should be borne in mind that any quantity whose exponent is 0 is equal to 1. (Art. 220.)

Multiply the following:

17. $a^2 - 2 + a^{-2}$ by $a^2 + 2 + a^{-2}$.

18. $a^{\frac{4}{3}} + a^{\frac{2}{3}}x^{\frac{2}{3}} + x^{\frac{4}{3}}$ by $a^{\frac{2}{3}} - x^{\frac{2}{3}}$.

19. $x^{-\frac{3}{4}} - x^{-\frac{1}{2}} + x^{-\frac{1}{4}} - 1$ by $x^{-\frac{1}{4}} + 1$.

20. $x^{-2} - 2x^{-1} + 1 - 2x$ by $x^{-3} + 2x^{-2}$.

21. $3a^{-1} - a^{-2}b^{-1} + a^{-3}b^{-2}$ by $6a^3b^2 + 2a^2b + 2a$.

22. $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 4 + x^{-\frac{1}{3}}$ by $3x^{\frac{4}{3}} + x - 2x^{\frac{2}{3}}$.

23. $x^{-3}y^2 - x^{-2}y - 2x^{-1}$ by $2x^2y^{-1} + 2x^3y^{-2} - 4x^4y^{-3}$.

24. $a^{\frac{2}{3}}x^{-\frac{3}{4}} + 2 + a^{-\frac{2}{3}}x^{\frac{3}{4}}$ by $2a^{-\frac{2}{3}}x^{\frac{3}{4}} - 4a^{-\frac{4}{3}}x^{\frac{3}{2}} + 2a^{-2}x^{\frac{9}{4}}$.

25. $3a^{\frac{3}{4}}b^{-1} + a^{\frac{1}{4}} - 2a^{-\frac{1}{4}}b$ by $6a^{\frac{1}{4}}b^{-1} - 2a^{-\frac{1}{4}} - 3a^{-\frac{3}{4}}b$.

225. The rule of Art. 89 for the division of exponents holds universally; for, it follows from Art. 222 that

$$\frac{a^m}{a^n} = a^m \times a^{-n} = a^{m-n}. \qquad \text{(Art. 224.)}$$

For example, $\dfrac{a^{-\frac{3}{4}}}{a} = a^{-\frac{3}{4}-1} = a^{-\frac{7}{4}};$

$$\frac{a^{\frac{1}{2}}}{a^{-2}} = a^{\frac{1}{2}+2} = a^{\frac{5}{2}};$$

$$\frac{a^{-3}}{a^{-\frac{2}{5}}} = a^{-3+\frac{2}{5}} = a^{-\frac{13}{5}}; \text{ etc.}$$

EXAMPLES.

Divide the following:

1. a^3 by a^{-1}.
2. a by a^3.
3. $a^{\frac{3}{7}}$ by $a^{\frac{4}{5}}$.
4. $a^{-\frac{1}{2}}$ by $a^{-\frac{4}{7}}$.
5. $3c^{-1}$ by $\sqrt[4]{c^5}$.
6. m^2 by $\sqrt[5]{m^{-2}}$.
7. $x^{\frac{1}{3}}$ by $\dfrac{1}{\sqrt[4]{x^3}}$.
8. $15a$ by $3a^{-1}\sqrt[3]{b}$.
9. $6x^{-1}y^{\frac{2}{3}}$ by $3xy^{-\frac{1}{5}}$.

10. Divide $2a^{\frac{2}{3}} - 20 + 18a^{-\frac{2}{3}}$ by $a + 2a^{\frac{2}{3}} - 3a^{\frac{1}{3}}$.

$$
\begin{array}{l|l}
2a^{\frac{2}{3}} - 20 + 18a^{-\frac{2}{3}} & a + 2a^{\frac{2}{3}} - 3a^{\frac{1}{3}} \\ \cline{2-2}
\underline{2a^{\frac{2}{3}} + 4a^{\frac{1}{3}} - 6} & 2a^{-\frac{1}{3}} - 4a^{-\frac{2}{3}} - 6a^{-1}, \textit{ Ans.} \\
\quad - 4a^{\frac{1}{3}} - 14 + 18a^{-\frac{2}{3}} & \\
\quad \underline{- 4a^{\frac{1}{3}} - 8 + 12a^{-\frac{1}{3}}} & \\
\qquad\qquad - 6 - 12a^{-\frac{1}{3}} + 18a^{-\frac{2}{3}} & \\
\qquad\qquad \underline{- 6 - 12a^{-\frac{1}{3}} + 18a^{-\frac{2}{3}}} &
\end{array}
$$

Note. It is important to arrange the dividend and divisor in the same order of powers, and to keep this order throughout the work.

Divide the following:

11. $a - b$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$. **12.** $a^{-2} + 1$ by $a^{-\frac{2}{3}} + 1$.

13. $x^{-2} - 5x^{-1} - 46 - 40x$ by $x^{-1} + 4$.

14. $x^{-3} - 1$ by $x^{-\frac{9}{4}} - x^{-\frac{3}{2}} + x^{-\frac{3}{4}} - 1$.

15. $m - 3m^{\frac{2}{3}}n^{\frac{1}{3}} + 3m^{\frac{1}{3}}n^{\frac{2}{3}} - n$ by $m^{\frac{1}{3}} - n^{\frac{1}{3}}$.

16. $x^{-3}y^{-5} - 3x^{-5}y^{-7} + x^{-7}y^{-9}$ by $x^{-2}y^{-3} + x^{-3}y^{-4} - x^{-4}y^{-5}$.

17. $a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b$ by $a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$.

18. $m^{-\frac{4}{3}} + m^{-\frac{2}{3}}n^{-2} + n^{-4}$ by $m^{-1} + m^{-\frac{2}{3}}n^{-1} + m^{-\frac{1}{3}}n^{-2}$.

226. We will now prove that Rule II., Art. 216, holds for all values of m and n.

We will consider three cases, in each of which m may have any value, positive or negative, integral or fractional.

Case I. Let n be a positive integer.

Then, from the definition of a positive integral exponent,

$$
\begin{aligned}
(a^m)^n &= a^m \times a^m \times a^m \times \cdots \text{ to } n \text{ factors} \\
&= a^{m+m+m+\cdots \text{ to } n \text{ terms}} = a^{mn}.
\end{aligned}
$$

CASE II. Let n be a positive fraction, which we will denote by $\frac{p}{q}$.

Then, $(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p}$, by the definition of Art. 218,

$= \sqrt[q]{a^{mp}}$, by Case I.,

$= a^{\frac{mp}{q}}$, by Art. 218,

$= a^{m\times\frac{p}{q}}$.

CASE III. Let n be a negative quantity, which we will denote by $-s$.

Then, $(a^m)^{-s} = \frac{1}{(a^m)^s}$, by the definition of Art. 221,

$= \frac{1}{a^{ms}}$, by Cases I. or II.,

$= a^{-ms} = a^{m(-s)}$.

We have therefore for all values of m and n,

$$(a^m)^n = a^{mn}.$$

For example,

$$(a^{-\frac{2}{3}})^{\frac{1}{2}} = a^{-\frac{2}{3}\times\frac{1}{2}} = a^{-\frac{1}{3}};$$

$$(a^2)^{-3} = a^{2\times-3} = a^{-6};$$

$$(a^{-3})^{-\frac{1}{3}} = a^{-3\times-\frac{1}{3}} = a;\ \text{etc.}$$

EXAMPLES.

227. Find the values of the following:

1. $(a^2)^{-3}$.
2. $(a^{-2})^2$.
3. $(a^3)^{\frac{5}{2}}$.
4. $(c^{-\frac{2}{5}})^{\frac{10}{3}}$.
5. $(x^{-\frac{3}{4}})^{-2}$.
6. $(a^{-1})^{\frac{1}{3}}$.
7. $(a^{\frac{1}{3}})^{\frac{6}{5}}$.
8. $(\sqrt{x})^{-\frac{1}{3}}$.
9. $(\sqrt[4]{m^3})^{\frac{2}{3}}$.
10. $(\sqrt[5]{y^{-3}})^{-5}$.
11. $\left(\frac{1}{a^2}\right)^{\frac{3}{2}}$.
12. $(x^{\frac{2n}{3}})^{-\frac{6}{n}}$.
13. $\left(\frac{1}{\sqrt{c}}\right)^{\frac{2}{5}}$.
14. $\left(\frac{1}{\sqrt[4]{n^3}}\right)^{\frac{4}{3}}$.
15. $\sqrt[3]{[(x^{-\frac{1}{2}})^2]}$.
16. $(a^{1-\frac{n}{m}})^{\frac{1}{m-n}}$.

228. *To prove that* $(ab)^n = a^n b^n$, *for any value of* n.

In Art. 193 we showed the truth of the theorem for a positive integral value of n.

Case I. Let n be a positive fraction, which we will denote by $\frac{p}{q}$.

By Art. 226, $\qquad [(ab)^{\frac{p}{q}}]^q = (ab)^p.$

By Art. 193, $\qquad [a^{\frac{p}{q}} b^{\frac{p}{q}}]^q = a^p b^p = (ab)^p.$

Therefore, $\qquad [(ab)^{\frac{p}{q}}]^q = [a^{\frac{p}{q}} b^{\frac{p}{q}}]^q.$

Taking the qth root of both members,

$$(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}.$$

Case II. Let n be a negative quantity, which we will denote by $-s$.

Then, $\qquad (ab)^{-s} = \dfrac{1}{(ab)^s} = \dfrac{1}{a^s b^s}$, by Art. 193, or Case I.

$$= a^{-s} b^{-s}.$$

MISCELLANEOUS EXAMPLES.

229. Square the following (Art. 95):

1. $a^{\frac{2}{3}} - b^{\frac{1}{2}}$. $\qquad$ 2. $a^{-\frac{3}{2}} + 2a$. $\qquad$ 3. $x^{-1}y^2 - 3x^2y^{-3}$.

Extract the square roots of the following:

4. $a^{-2}x^{\frac{3}{4}}$. $\qquad$ 5. $9mn^{\frac{1}{3}}$. $\qquad$ 6. $\dfrac{c^{\frac{2}{3}} d^{-\frac{5}{4}}}{4xy^3}$. $\qquad$ 7. $\dfrac{a^{-\frac{2}{3}} b^{-1}}{cd^4 e^{\frac{1}{2}}}$.

8. $9x^{-4} - 12x^{-3} - 2x^{-2} + 4x^{-1} + 1$.

9. $4x^{\frac{4}{3}} + 4x^{\frac{5}{3}} - 15x^2 - 8x^{\frac{7}{3}} + 16x^{\frac{8}{3}}$.

10. $a^3b^{-\frac{2}{3}} - 4a^{\frac{3}{2}}b^{-\frac{1}{3}} + 6 - 4a^{-\frac{3}{2}}b^{\frac{1}{3}} + a^{-3}b^{\frac{2}{3}}$.

Extract the cube roots of the following:

11. ab^2. **12.** $-8x^{-4}y^{\frac{2}{3}}$. **13.** $27m^2n^{-\frac{3}{7}}$. **14.** $\dfrac{a^{-1}b}{64x^{\frac{1}{3}}}$.

15. $8y^{-2}-12y^{-\frac{11}{6}}+6y^{-\frac{5}{3}}-y^{-\frac{3}{2}}$.

16. $x^{\frac{1}{2}}-9x^{\frac{2}{3}}+33x^{\frac{5}{6}}-63x+66x^{\frac{7}{6}}-36x^{\frac{4}{3}}+8x^{\frac{3}{2}}$.

Reduce the following to their simplest forms:

17. $a^{x-y+2z}a^{2x+y-3z}a^{z}$.

18. $\dfrac{x^{m+n}x^{m+r}x^{r-m}}{x^{n+2m-r}}$.

19. $(x^{a})^{-b}\div(x^{-a})^{-b}$.

20. $\left[x^{a^2-ab}x^{b^2-ab}\right]^{\frac{1}{a-b}}$.

21. $\left(\dfrac{a^{x+y}}{a^{y}}\right)^{x}\div\left(\dfrac{a^{y}}{a^{y-x}}\right)^{x-y}$.

22. $\left[\left(x^{\frac{1}{a-b}}\right)^{a-\frac{b^2}{a}}\right]^{\frac{a}{a+b}}$.

23. $\dfrac{x^{\frac{1}{2}}(a^{\frac{1}{2}}-x^{\frac{1}{2}})-x^{\frac{3}{2}}(a^{-\frac{1}{2}}-x^{-\frac{1}{2}})}{2(a^{\frac{1}{2}}+x^{\frac{1}{2}})(a^{\frac{1}{2}}-x^{\frac{1}{2}})}$.

24. $\dfrac{a-b}{a^{\frac{1}{4}}b^{-\frac{1}{4}}+a^{-\frac{1}{4}}b^{\frac{1}{4}}}$.

25. $\dfrac{(1-a^{\frac{1}{2}}x^{\frac{1}{2}})^2+(x^{\frac{1}{2}}+a^{\frac{1}{2}})^2}{1-a^{\frac{1}{2}}x^{\frac{1}{2}}+a^{\frac{1}{2}}(a^{\frac{1}{2}}+x^{\frac{1}{2}})}$

26. $(4x^3-3x)(x^2+1)^{-\frac{1}{3}}+3x(x^2+1)^{\frac{2}{3}}$.

27. $\dfrac{(1-3x+x^2)^{\frac{1}{2}}-x(x-3)(1-3x+x^2)^{-\frac{1}{2}}}{1-3x+x^2}$.

28. $\dfrac{m[x^{-\frac{1}{2}}+(m+x)^{-\frac{1}{2}}]}{2[x^{\frac{1}{2}}+(m+x)^{\frac{1}{2}}]}+\dfrac{m+2x}{2x^{\frac{1}{2}}(m+x)^{\frac{1}{2}}}$.

29. $\dfrac{x^2+[1+(1+x^2)^{\frac{1}{2}}]^2}{2[1+(1+x^2)^{\frac{1}{2}}]}$.

XX. RADICALS.

230. A **Radical** is a root of a quantity indicated by a radical sign; as, $\sqrt{a}$, or $\sqrt[3]{x+1}$.

If the indicated root can be exactly obtained, it is called a *rational* quantity; if it cannot be exactly obtained, it is called an *irrational* or *surd* quantity.

231. The *degree* of a radical is denoted by the index of the radical sign; thus, $\sqrt[3]{x+1}$ is of the *third* degree.

232. *Similar Radicals* are those of the same degree, and with the same quantity under the radical sign; as, $2\sqrt[5]{ax}$ and $3\sqrt[5]{ax}$.

233. Most problems in radicals depend for their solution on the following important principle (Art. 228):

For any value of n, $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}$.

That is, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

TO REDUCE A RADICAL TO ITS SIMPLEST FORM.

234. A radical is in its *simplest form* when the quantity under the radical sign is not a perfect power of the degree denoted by any factor of the index of the radical, and has no factor which is a perfect power of the same degree as the radical.

CASE I.

235. *When the quantity under the radical sign is a perfect power of the degree denoted by a factor of the index.*

1. Reduce $\sqrt[6]{8}$ to its simplest form.

$$\sqrt[6]{8} = \sqrt[6]{2^3} = 2^{\frac{3}{6}} = 2^{\frac{1}{2}} = \sqrt{2}, \text{ \textit{Ans.}}$$

EXAMPLES.

Reduce the following to their simplest forms:

2. $\sqrt[4]{25}$.	5. $\sqrt[12]{27}$.	8. $\sqrt[12]{64}$.	11. $\sqrt[4]{49m^4n^6}$.
3. $\sqrt[6]{9}$.	6. $\sqrt[10]{100}$.	9. $\sqrt[8]{25x^4}$.	12. $\sqrt[6]{125a^3b^9}$.
4. $\sqrt[9]{8}$.	7. $\sqrt[12]{81}$.	10. $\sqrt[10]{32a^5}$.	13. $\sqrt[mn]{a^nb^{2n}}$.

Case II.

236. *When the quantity under the radical sign has a factor which is a perfect power of the same degree as the radical.*

1. Reduce $\sqrt[3]{54}$ to its simplest form.

$\sqrt[3]{54} = \sqrt[3]{27 \times 2} = \sqrt[3]{27} \times \sqrt[3]{2}$ (Art. 233) $= 3\sqrt[3]{2}$, *Ans.*

2. Reduce $\sqrt{18a^2b^5 - 27a^3b^4}$ to its simplest form.

$$\begin{aligned}\sqrt{18a^2b^5 - 27a^3b^4} &= \sqrt{9a^2b^4(2b - 3a)} \\ &= \sqrt{9a^2b^4} \times \sqrt{2b - 3a} \\ &= 3ab^2\sqrt{2b - 3a}, \text{ Ans.}\end{aligned}$$

RULE.

Resolve the quantity under the radical sign into two factors, one of which is the highest perfect power of the same degree as the radical. Extract the required root of this factor, and prefix the result to the indicated root of the other.

Note. If the highest perfect power in the numerical portion of the quantity cannot be determined by inspection, it may be found by resolving the number into its prime factors.

Thus, $\sqrt{1944} = \sqrt{2^3 \times 3^5} = \sqrt{2^2 \times 3^4} \times \sqrt{2 \times 3}$

$= 2 \times 3^2 \times \sqrt{6} = 18\sqrt{6}.$

EXAMPLES.

Reduce the following to their simplest forms.

3. $\sqrt{50}$.
6. $\sqrt[3]{320}$.
9. $\sqrt[3]{81x^4y^3}$.
4. $3\sqrt{24}$.
7. $2\sqrt[4]{80}$.
10. $7\sqrt{63a^4b^5c^6}$.
5. $\sqrt{72}$.
8. $\sqrt{98a^3b^2}$.
11. $\sqrt[3]{250x^2y^3z^7}$.

12. $\sqrt{25x^3y^4-50x^4y^3}$.
14. $\sqrt{(x^2-y^2)(x+y)}$.
13. $\sqrt[3]{54a^4b^5+135a^3b^4}$.
15. $\sqrt{ax^2-6ax+9a}$.
16. $\sqrt{20x^2+60x+45}$.
17. $\sqrt{3m^3-54m^2n+243mn^2}$.

If the quantity under the radical sign is a fraction, *multiply both terms by such a quantity as will make the denominator a perfect power of the same degree as the radical.* Then proceed as before.

18. Reduce $\sqrt{\frac{9}{8a^3}}$ to its simplest form.

$$\sqrt{\frac{9}{8a^3}}=\sqrt{\frac{18a}{16a^4}}=\sqrt{\frac{9}{16a^4}\times 2a}=\frac{3}{4a^2}\sqrt{2a},\ \textit{Ans.}$$

Reduce the following to their simplest forms:

19. $\sqrt{\frac{3}{2}}$.
22. $\sqrt{\frac{4a^2}{27}}$.
25. $\frac{3}{11}\sqrt{\frac{4}{7}}$.
20. $\sqrt{\frac{5}{6}}$.
23. $\sqrt[3]{\frac{3x}{4}}$.
26. $\sqrt{\frac{9a^2b^3}{10cd}}$.
21. $\sqrt{\frac{7}{12}}$.
24. $\sqrt[3]{\frac{5}{9}}$.
27. $\sqrt{\frac{7xy^2}{8a^5}}$.
28. $\sqrt{\frac{ab^2}{4(a+x)}}$.
29. $\frac{a}{a^2-b^2}\sqrt{\frac{a^3c-2a^2bc+ab^2c}{b^3}}$.

237. *Conversely*, the coefficient of a radical may be introduced under the radical sign by raising it to the power denoted by the index.

1. Introduce the coefficient of $2a\sqrt[3]{3x^2}$ under the radical sign.

$$2a\sqrt[3]{3x^2}=\sqrt[3]{8a^3\times 3x^2}=\sqrt[3]{24a^3x^2},\ \textit{Ans.}$$

Note. A rational quantity may be expressed in the form of a radical by raising it to the power denoted by the index, and writing the result under the corresponding radical sign.

EXAMPLES.

Introduce the coefficients of the following under the radical signs:

2. $3\sqrt{5}$.
4. $3\sqrt[4]{2}$.
6. $4\sqrt{5ab}$.
8. $5a\sqrt[3]{2x^2}$.
3. $2\sqrt[3]{7}$.
5. $4\sqrt[3]{5}$.
7. $a^2b\sqrt[3]{ab^2}$.
9. $3mn^3\sqrt[4]{\dfrac{mn^2}{27}}$.
10. $(x-1)\sqrt{\dfrac{x+1}{x-1}}$.
12. $\dfrac{1+a}{1-a}\sqrt{\dfrac{1-a}{1+a}}$.
11. $(1+x)\sqrt{\dfrac{2}{1+x}-1}$.
13. $\dfrac{2x^2-1}{x}\sqrt{\dfrac{1}{(2x^2-1)^2}-1}$.

ADDITION AND SUBTRACTION OF RADICALS.

238. The sum or difference of two similar radicals (Art. 232) may be found by prefixing the sum or difference of their coefficients to their common radical part.

1. Find the sum of $\sqrt{20}$ and $\sqrt{45}$.

By Art 236, $\sqrt{20}=\sqrt{4\times 5}=2\sqrt{5}$,

$$\sqrt{45}=\sqrt{9\times 5}=3\sqrt{5}.$$

Hence, $\sqrt{20}+\sqrt{45}=2\sqrt{5}+3\sqrt{5}=5\sqrt{5}$, *Ans.*

2. Simplify $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}$.

$$\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}-\sqrt{\frac{9}{8}}=\sqrt{\frac{1}{4}\times 2}+\sqrt{\frac{1}{9}\times 6}-\sqrt{\frac{9}{16}\times 2}$$

$$=\frac{1}{2}\sqrt{2}+\frac{1}{3}\sqrt{6}-\frac{3}{4}\sqrt{2}$$

$$=\frac{1}{3}\sqrt{6}-\frac{1}{4}\sqrt{2}, \textit{ Ans.}$$

RULE.

Reduce each radical to its simplest form. Unite the similar radicals, and indicate the addition or subtraction of the dissimilar.

EXAMPLES.

Simplify the following:

3. $\sqrt{27}+\sqrt{12}$.

4. $\sqrt{96}+\sqrt{54}$.

5. $\sqrt{180}-\sqrt{45}$.

6. $\sqrt[3]{162}-\sqrt[3]{48}$.

7. $\sqrt{128}+\sqrt{98}+\sqrt{50}$.

8. $\sqrt{\frac{16}{15}}-\sqrt{\frac{3}{5}}$.

9. $\sqrt{4a^2b}+\sqrt{9b^3}$.

10. $\sqrt{75}+\sqrt{48}-\sqrt{245}$.

11. $\sqrt[3]{16}+\sqrt[3]{54}+\sqrt[3]{128}$.

12. $\sqrt{\frac{5}{9}}-\sqrt{\frac{1}{5}}+\sqrt{\frac{1}{45}}$.

13. $\sqrt{\frac{3}{8}}-\sqrt{\frac{1}{6}}+\sqrt{\frac{2}{27}}$.

14. $\sqrt[3]{\frac{1}{4}}+\sqrt[3]{\frac{1}{32}}+\sqrt[3]{\frac{2}{3}}$.

15. $7\sqrt{27}-\sqrt{75}-24\sqrt{\frac{1}{12}}-27\sqrt{\frac{1}{27}}$.

16. $\sqrt{27ab^2}+\sqrt{75a^3}+(a-3b)\sqrt{3a}$.

17. $\sqrt{9a^5+18a^4b}-\sqrt{4ab^6+8b^7}$.

18. $\sqrt[3]{24}+5\sqrt[3]{54}-\sqrt[3]{250}-\sqrt[3]{192}.$

19. $\sqrt{28a^2x-28ax+7x}-\sqrt{7a^2x+42ax+63x}.$

20. $x\sqrt{\dfrac{x-y}{x+y}}+y\sqrt{\dfrac{x+y}{x-y}}-\dfrac{3y^2-x^2}{x^2-y^2}\sqrt{x^2-y^2}.$

TO REDUCE RADICALS OF DIFFERENT DEGREES TO EQUIVALENT RADICALS OF THE SAME DEGREE.

239. **1.** Reduce $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ to equivalent radicals of the same degree.

By Art 218, $\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{6}{12}}=\sqrt[12]{2^6}=\sqrt[12]{64}.$

$$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{4}{12}}=\sqrt[12]{3^4}=\sqrt[12]{81}.$$

$$\sqrt[4]{5}=5^{\frac{1}{4}}=5^{\frac{3}{12}}=\sqrt[12]{5^3}=\sqrt[12]{125}.$$

RULE.

Write the radicals with fractional exponents, and reduce these exponents to a common denominator.

Note. The relative magnitude of radicals may be compared by reducing them to the same degree. Thus, in Ex. 1, $\sqrt[12]{125}$ is greater than $\sqrt[12]{81}$, and $\sqrt[12]{81}$ than $\sqrt[12]{64}$. Hence, $\sqrt[4]{5}$ is greater than $\sqrt[3]{3}$, and $\sqrt[3]{3}$ than $\sqrt{2}$.

EXAMPLES.

Reduce the following to equivalent radicals of the same degree:

2. $\sqrt[3]{2}$ and $\sqrt{3}$.

3. $\sqrt{5}$, $\sqrt[3]{4}$, and $\sqrt[3]{3}$.

4. $\sqrt[3]{5}$, $\sqrt[4]{6}$, and $\sqrt[6]{7}$.

5. $\sqrt[3]{2a}$, $\sqrt[5]{3b}$, and $\sqrt[5]{4c}$.

6. $\sqrt[6]{xy}$, $\sqrt[4]{yz}$, and $\sqrt[8]{zx}$.

7. $\sqrt[6]{a+b}$ and $\sqrt[4]{a-b}$.

8. Which is the greater, $\sqrt[3]{2}$ or $\sqrt[5]{3}$?

9. Which is the greater, $\sqrt[4]{3}$ or $\sqrt[5]{5}$?

10. Arrange in order of magnitude $\sqrt{3}$, $\sqrt[3]{4}$, and $\sqrt[4]{7}$.

MULTIPLICATION OF RADICALS.

240. 1. Multiply $\sqrt{6}$ by $\sqrt{15}$.

By Art. 233,

$$\sqrt{6}\times\sqrt{15}=\sqrt{6\times 15}=\sqrt{90}=3\sqrt{10},\ \textit{Ans.}$$

2. Multiply $\sqrt{2a}$ by $\sqrt[3]{3a^2}$.

Reducing to equivalent radicals of the same degree,

$$\sqrt{2a}=(2a)^{\frac{1}{2}}=(2a)^{\frac{3}{6}}=\sqrt[6]{(2a)^3}=\sqrt[6]{8a^3}$$

$$\sqrt[3]{3a^2}=(3a^2)^{\frac{1}{3}}=(3a^2)^{\frac{2}{6}}=\sqrt[6]{(3a^2)^2}=\sqrt[6]{9a^4}$$

Hence, $\sqrt{2a}\times\sqrt[3]{3a^2}=\sqrt[6]{8a^3}\times\sqrt[6]{9a^4}=\sqrt[6]{72a^7}$

$$=a\sqrt[6]{72a},\ \textit{Ans.}$$

RULE.

Reduce the radicals to equivalent radicals of the same degree. Multiply together the quantities under the radical signs, and write the product under the common radical sign.

Note. The result should be reduced to its simplest form.

EXAMPLES.

Multiply the following:

3. $\sqrt{6}$ and $\sqrt{42}$.

4. $5\sqrt{10}$ and $3\sqrt{15}$.

5. $2\sqrt{3x}$ and $5\sqrt{15x}$.

6. $\sqrt[3]{a^2b}$ and $\sqrt[3]{abc^2}$.

7. $\frac{3}{4}\sqrt[3]{12}$ and $\frac{2}{3}\sqrt[3]{2}$.

8. $\sqrt[3]{2}$ and $\sqrt[4]{3}$.

9. $\sqrt{ax}$ and $\sqrt[3]{bx}$.

10. $\sqrt[3]{4a^2}$ and $\sqrt{2a}$.

11. $4\sqrt[5]{3}$ and $3\sqrt{2}$.

12. $\sqrt[4]{xy}$, $\sqrt[4]{yz}$, and $\sqrt[4]{zx}$.

13. $\sqrt{3}$, $\sqrt[3]{2}$, and $\sqrt[3]{\frac{1}{6}}$.

14. $\sqrt[5]{2x}$, $\sqrt[3]{3x}$, and $\sqrt[5]{\frac{1}{3x^2}}$.

15. Multiply $2\sqrt{3}+3\sqrt{2}$ by $3\sqrt{3}-\sqrt{2}$.

$$\begin{array}{l}
2\sqrt{3}+3\sqrt{2} \\
3\sqrt{3}-\ \ \sqrt{2} \\
\hline
18+9\sqrt{6} \\
\quad\quad -2\sqrt{6}-6 \\
\hline
18+7\sqrt{6}-6=12+7\sqrt{6},\ \textit{Ans.}
\end{array}$$

Note. It should be remembered that to multiply a radical of the second degree by itself simply removes the radical sign; thus, $\sqrt{3}\times\sqrt{3}=3$.

16. Multiply $3\sqrt{x^2+1}+4x$ by $2\sqrt{x^2+1}-x$.

$$\begin{array}{l}
3\sqrt{x^2+1}+4x \\
2\sqrt{x^2+1}-\ \ x \\
\hline
6(x^2+1)+8x\sqrt{x^2+1} \\
\quad\quad\quad\quad -3x\sqrt{x^2+1}-4x^2 \\
\hline
6x^2+6+5x\sqrt{x^2+1}-4x^2 \\
\quad\quad\quad =2x^2+6+5x\sqrt{x^2+1},\ \textit{Ans.}
\end{array}$$

Multiply the following:

17. $\sqrt{x}-2$ and $\sqrt{x}+3$.

18. $\sqrt{5}-3\sqrt{2}$ and $2\sqrt{5}+\sqrt{2}$.

19. $\sqrt{x}-4\sqrt{3}$ and $2\sqrt{x}+\sqrt{3}$.

20. $2\sqrt{a}-3\sqrt{b}$ and $4\sqrt{a}+\sqrt{b}$.

21. $\sqrt{x}-\sqrt{y}+\sqrt{z}$ and $\sqrt{x}+\sqrt{y}-\sqrt{z}$.

22. $\sqrt{x+1}-2\sqrt{x}$ and $2\sqrt{x+1}+\sqrt{x}$.

23. $\sqrt{2}-\sqrt{3}+\sqrt{5}$ and $\sqrt{2}+\sqrt{3}+\sqrt{5}$.

24. $3\sqrt{5}-2\sqrt{6}+\sqrt{7}$ and $6\sqrt{5}+4\sqrt{6}-2\sqrt{7}$.

25. $8\sqrt{3}+10\sqrt{2}-3\sqrt{5}$ and $4\sqrt{3}-5\sqrt{2}-\sqrt{5}$.

Expand the following (Art. 95):

26. $(2\sqrt{3}-3)^2$.

27. $(3\sqrt{8}+5\sqrt{3})^2$.

28. $(\sqrt{1-a^2}+a)^2$.

29. $(\sqrt{a+b}-\sqrt{a-b})^2$.

30. $(\sqrt{x^2+1}+x)(\sqrt{x^2+1}-x)$.

31. $(\sqrt{x+1}+\sqrt{x-1})(\sqrt{x+1}-\sqrt{x-1})$.

32. $(3\sqrt{2x+5}+2\sqrt{3x-1})(3\sqrt{2x+5}-2\sqrt{3x-1})$.

DIVISION OF RADICALS.

241. By Art. 233, $\sqrt[n]{ab}=\sqrt[n]{a}\times\sqrt[n]{b}$.

Whence, $$\frac{\sqrt[n]{ab}}{\sqrt[n]{a}}=\sqrt[n]{b}.$$

RULE.

Reduce the radicals to equivalent radicals of the same degree. Divide the quantities under the radical sign, and write the quotient under the common radical sign.

EXAMPLES.

1. Divide $\sqrt[3]{15}$ by $\sqrt{5}$.

Reducing to equivalent radicals of the same degree, we have

$$\frac{\sqrt[3]{15}}{\sqrt{5}}=\frac{\sqrt[6]{225}}{\sqrt[6]{125}}=\sqrt[6]{\frac{225}{125}}=\sqrt[6]{\frac{9}{5}},\ \textit{Ans.}$$

Divide the following:

2. $\sqrt{108}$ by $\sqrt{6}$.
3. $\sqrt{50c^3}$ by $\sqrt{2c}$.
4. $\sqrt[3]{9a^4}$ by $\sqrt[3]{3a}$.
5. $\sqrt{6}$ by $\sqrt[3]{3}$.
6. $\sqrt[3]{18}$ by $\sqrt{6}$.
7. $\sqrt[5]{2}$ by $\sqrt[4]{3}$.
8. $\sqrt[5]{12}$ by $\sqrt[3]{2}$.
9. $\sqrt[3]{4a}$ by $\sqrt[4]{2a}$.
10. $\sqrt[3]{3a^2b}$ by $\sqrt[5]{6a^3b^2}$.
11. $\sqrt[6]{12x^3y^2z^5}$ by $\sqrt[4]{2x^2yz^3}$.

INVOLUTION AND EVOLUTION OF RADICALS.

242. Any power or root of a radical may be found by using fractional exponents.

1. Raise $\sqrt[6]{12}$ to the third power.

$$(\sqrt[6]{12})^3 = (12^{\frac{1}{6}})^3 = 12^{\frac{3}{6}} = 12^{\frac{1}{2}} = \sqrt{12} = 2\sqrt{3}, \textit{ Ans.}$$

2. Raise $\sqrt[3]{2}$ to the fourth power.

$$(\sqrt[3]{2})^4 = (2^{\frac{1}{3}})^4 = 2^{\frac{4}{3}} = \sqrt[3]{2^4} = \sqrt[3]{16} = 2\sqrt[3]{2}, \textit{ Ans.}$$

Note 1. The following rule for the involution of radicals is evident from the above:

If possible, divide the index by the exponent of the required power; otherwise, raise the quantity under the radical sign to the required power.

EXAMPLES.

Find the values of the following:

3. $(\sqrt[5]{5})^3$.
4. $(\sqrt[4]{7})^2$.
5. $(\sqrt[3]{a^2x})^5$.
6. $(\sqrt[6]{18})^3$.
7. $(\sqrt[8]{a-b})^4$.
8. $(4\sqrt{3x})^3$.
9. $(\sqrt[12]{32})^3$.
10. $(3a\sqrt[3]{bx})^4$.
11. $(3\sqrt[6]{24a^4b^6})^2$.

12. Extract the cube root of $\sqrt{27x^3}$.

$$\sqrt[3]{(\sqrt{27x^3})} = (\sqrt{27x^3})^{\frac{1}{3}} = (\sqrt{(3x)^3})^{\frac{1}{3}} = [(3x)^{\frac{3}{2}}]^{\frac{1}{3}}$$
$$= (3x)^{\frac{1}{2}} = \sqrt{3x}, \textit{ Ans.}$$

13. Extract the square root of $\sqrt[3]{6}$.

$$\sqrt{(\sqrt[3]{6})} = (6^{\frac{1}{3}})^{\frac{1}{2}} = 6^{\frac{1}{6}} = \sqrt[6]{6}, \textit{ Ans.}$$

Note 2. The following rule for the evolution of radicals is evident from the above:

If possible, extract the required root of the quantity under the radical sign; otherwise, multiply the index of the radical by the index of the required root.

If the radical has a coefficient which is not a perfect power of the same degree as the required root, it should be introduced under the radical sign before applying the rule. Thus,

$$\sqrt[3]{2\sqrt{2}} = \sqrt[3]{\sqrt{8}} = \sqrt{2}.$$

Find the values of the following:

14. $\sqrt{(\sqrt{2})}$.

15. $\sqrt[3]{(\sqrt{125})}$.

16. $\sqrt[5]{(\sqrt{32})}$.

17. $\sqrt[3]{(\sqrt[5]{27a^3})}$.

18. $\sqrt[3]{(\sqrt[4]{a+b})}$.

19. $\sqrt{(\sqrt[3]{x^2-2x+1})}$.

20. $\sqrt[3]{(3\sqrt{3})}$.

21. $\sqrt[4]{(\sqrt[5]{x^8y^{12}})}$.

22. $\sqrt[5]{(4\sqrt{2})}$.

TO REDUCE A FRACTION HAVING AN IRRATIONAL DENOMINATOR TO AN EQUIVALENT FRACTION WHOSE DENOMINATOR IS RATIONAL.

Case I.

243. *When the denominator is a monomial.*

The reduction is effected by multiplying both terms by a radical of the same degree as the denominator, having such a quantity under the radical sign as will make the denominator of the resulting fraction rational.

1. Reduce $\frac{5}{\sqrt[3]{3a^2}}$ to an equivalent fraction with a rational denominator.

Multiplying both terms by $\sqrt[3]{9a}$,

$$\frac{5}{\sqrt[3]{3a^2}}=\frac{5\sqrt[3]{9a}}{\sqrt[3]{3a^2}\sqrt[3]{9a}}=\frac{5\sqrt[3]{9a}}{\sqrt[3]{27a^3}}=\frac{5\sqrt[3]{9a}}{3a},\ \textit{Ans.}$$

EXAMPLES.

Reduce the following to equivalent fractions with rational denominators:

2. $\frac{3}{\sqrt{2}}$.

3. $\frac{1}{\sqrt[3]{4}}$.

4. $\frac{2x}{\sqrt{5xy}}$.

5. $\frac{5}{\sqrt[3]{9a^2}}$.

6. $\frac{1}{\sqrt[5]{16x^3}}$.

7. $\frac{2c}{\sqrt[4]{27a^2}}$.

Case II.

244. *When the denominator is a binomial, containing radicals of the second degree only.*

1. Reduce $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ to an equivalent fraction with a rational denominator.

Multiplying both terms by $\sqrt{5}-\sqrt{2}$,

$$\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{(\sqrt{5}-\sqrt{2})^2}{(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})}=\frac{5-2\sqrt{10}+2}{5-2}$$

$$=\frac{7-2\sqrt{10}}{3},\ \textit{Ans.}$$

2. Reduce $\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}$ to an equivalent fraction with a rational denominator.

Multiplying both terms by $1+\sqrt{1-x}$,

$$\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}=\frac{(1+\sqrt{1-x})^2}{(1-\sqrt{1-x})(1+\sqrt{1-x})}$$

$$=\frac{1+2\sqrt{1-x}+1-x}{1-(1-x)}$$

$$=\frac{2-x+2\sqrt{1-x}}{x}, \text{ } Ans.$$

RULE.

Multiply both terms of the fraction by the denominator with the sign between its terms changed.

EXAMPLES.

Reduce the following to equivalent fractions with rational denominators:

3. $\frac{4}{3+\sqrt{2}}$.

4. $\frac{4-\sqrt{3}}{2-\sqrt{3}}$.

5. $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$.

6. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

7. $\frac{2\sqrt{5}+\sqrt{2}}{\sqrt{5}-3\sqrt{2}}$.

8. $\frac{a-\sqrt{x}}{a+\sqrt{x}}$.

9. $\frac{\sqrt{a+1}-2}{\sqrt{a+1}-1}$.

10. $\frac{\sqrt{x+2}-\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}$.

11. $\frac{a-\sqrt{a^2-1}}{a+\sqrt{a^2-1}}$.

12. $\frac{x+\sqrt{x^2-4}}{x-\sqrt{x^2-4}}$.

13. $\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$.

14. $\frac{\sqrt{a^2-1}-\sqrt{a^2+1}}{\sqrt{a^2-1}+\sqrt{a^2+1}}$.

245. The approximate value of a fraction, whose denominator is irrational, may be most conveniently found by reducing it to an equivalent fraction with a rational denominator.

1. Find the approximate value of $\frac{1}{2-\sqrt{2}}$ to three places of decimals.

$$\frac{1}{2-\sqrt{2}}=\frac{2+\sqrt{2}}{(2-\sqrt{2})(2+\sqrt{2})}$$

$$=\frac{2+\sqrt{2}}{4-2}$$

$$=\frac{2+1.414...}{2}=1.707..., \textit{Ans.}$$

EXAMPLES.

Find the approximate values of the following to three decimal places:

2. $\frac{3}{\sqrt{2}-1}$. 3. $\frac{7}{\sqrt[3]{9}}$. 4. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$. 5. $\frac{2\sqrt{5}-\sqrt{3}}{3\sqrt{5}+2\sqrt{3}}$.

IMAGINARY QUANTITIES.

246. An **Imaginary Quantity** is an indicated even root of a negative quantity (Art. 201); as, $\sqrt{-4}$, or $\sqrt[4]{-a^2}$.

In contradistinction, all other quantities, rational or irrational, are called *real* quantities.

247. Every imaginary square root can be expressed as the product of a real quantity multiplied by $\sqrt{-1}$. Thus,

$$\sqrt{-a^2}=\sqrt{a^2\times(-1)}=\sqrt{a^2}\times\sqrt{-1}=a\sqrt{-1};$$

$$\sqrt{-5}=\sqrt{5\times(-1)}=\sqrt{5}\sqrt{-1}; \text{ etc.}$$

248. Let it be required to find the powers of $\sqrt{-1}$.

By Art. 198, $\sqrt{-1}$ signifies a quantity which, when multiplied by itself, will produce -1; that is,

$$(\sqrt{-1})^2=-1.$$

Therefore,

$$(\sqrt{-1})^3=(\sqrt{-1})^2\times\ \sqrt{-1}\ =(-1)\times\sqrt{-1}=-\sqrt{-1};$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2\times(\sqrt{-1})^2=(-1)\times(-1)=1;$$
$$(\sqrt{-1})^5=(\sqrt{-1})^4\times\ \sqrt{-1}\ =\ \ 1\ \times\sqrt{-1}=\sqrt{-1};\text{ etc.}$$

Thus the first four powers of $\sqrt{-1}$ are $\sqrt{-1}$, -1, $-\sqrt{-1}$, and 1; and for higher powers these terms recur in the same order.

MULTIPLICATION OF IMAGINARY QUANTITIES.

249. The product of two or more imaginary square roots may be found by aid of the principles of Arts. 247 and 248.

1. Multiply $\sqrt{-2}$ by $\sqrt{-3}$.

By Art. 247,

$$\begin{aligned}\sqrt{-2}\times\sqrt{-3}&=\sqrt{2}\sqrt{-1}\times\sqrt{3}\sqrt{-1}\\&=\sqrt{2}\sqrt{3}(\sqrt{-1})^2\\&=\sqrt{6}\times(-1)\ (\text{Art. }248)=-\sqrt{6},\ \textit{Ans.}\end{aligned}$$

2. Multiply $\sqrt{-a^2}$, $\sqrt{-b^2}$, and $\sqrt{-c^2}$.

$$\begin{aligned}\sqrt{-a^2}\times\sqrt{-b^2}\times\sqrt{-c^2}&=a\sqrt{-1}\times b\sqrt{-1}\times c\sqrt{-1}\\&=abc(\sqrt{-1})^3=-abc\sqrt{-1},\ \textit{Ans.}\end{aligned}$$

RULE.

Reduce each imaginary quantity to the form of a real quantity multiplied by $\sqrt{-1}$. Form the product of the real quantities, and multiply the result by the required power of $\sqrt{-1}$.

EXAMPLES.

Multiply the following:

3. $4\sqrt{-3}$ and $2\sqrt{-2}$.
4. $\sqrt{-a^2}$ and $\sqrt{-x^2}$.
5. $-3\sqrt{-a}$ and $4\sqrt{-b}$.
6. $\sqrt{-3}$, $\sqrt{-4}$, and $\sqrt{-5}$.
7. $1-2\sqrt{-1}$ and $3+\sqrt{-1}$.
8. $4+\sqrt{-7}$ and $8-2\sqrt{-7}$.
9. $2\sqrt{-3}-3\sqrt{-2}$ and $4\sqrt{-3}+6\sqrt{-2}$.
10. $\sqrt{-1}$, $\sqrt{-9}$, $\sqrt{-16}$, and $\sqrt{-25}$.

Expand the following:

11. $(2-\sqrt{-3})^2$.
12. $(\sqrt{-3}+2\sqrt{-2})^2$.
13. $(1+\sqrt{-1})(1-\sqrt{-1})$.
14. $(a+\sqrt{-b})(a-\sqrt{-b})$.
15. $(x\sqrt{-x}+y\sqrt{-y})(x\sqrt{-x}-y\sqrt{-y})$.
16. $(1+\sqrt{-1})^2+(1-\sqrt{-1})^2$.
17. Divide $\sqrt{-x}$ by $\sqrt{-y}$.

$$\frac{\sqrt{-x}}{\sqrt{-y}}=\frac{\sqrt{x}\sqrt{-1}}{\sqrt{y}\sqrt{-1}}=\frac{\sqrt{x}}{\sqrt{y}}=\sqrt{\frac{x}{y}},\ \textit{Ans.}$$

Note. The rule of Art. 241 would have given the same result; hence, that rule applies to the division of all radicals, whether real or imaginary.

Divide the following:

18. $\sqrt{-6}$ by $\sqrt{-2}$.
19. $\sqrt{-24}$ by $\sqrt{-3}$.
20. $\sqrt[4]{-12}$ by $\sqrt[4]{-3}$.
21. $\sqrt[6]{-54}$ by $\sqrt[6]{-2}$.

PROPERTIES OF QUADRATIC SURDS.

250. A **Quadratic Surd** is the indicated square root of an imperfect square; as, $\sqrt{3}$, or $\sqrt{7}$.

251. *A quadratic surd cannot be equal to a rational quantity plus a quadratic surd.*

For, if possible, let $\sqrt{a} = b + \sqrt{c}$.

Squaring the equation, $a = b^2 + 2b\sqrt{c} + c$.

Or, $2b\sqrt{c} = a - b^2 - c$.

Whence, $\sqrt{c} = \dfrac{a - b^2 - c}{2b}$.

That is, a surd equal to a rational quantity, which is impossible. Hence $\sqrt{a}$ cannot be equal to $b + \sqrt{c}$.

252. *To prove that if $a + \sqrt{b} = c + \sqrt{d}$, then $a = c$, and $\sqrt{b} = \sqrt{d}$.*

If a is not equal to c, let $a = c + x$. Substituting, we have

$$c + x + \sqrt{b} = c + \sqrt{d}.$$

Or, $x + \sqrt{b} = \sqrt{d}$,

which is impossible by Art. 251. Hence $a = c$, and consequently $\sqrt{b} = \sqrt{d}$.

253. *To prove that if $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, then $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.*

Squaring the equation $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$,

we have $a + \sqrt{b} = x + 2\sqrt{xy} + y$.

Whence, by Art. 252, $a = x + y$, (1)

and $\sqrt{b} = 2\sqrt{xy}$. (2)

Subtracting (2) from (1), $a - \sqrt{b} = x - 2\sqrt{xy} + y$.

Extracting the square root, $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

SQUARE ROOT OF A BINOMIAL SURD.

254. The preceding principles serve to extract the square root of a binomial surd whose first term is rational.

For example, required the square root of $13 - \sqrt{160}$.

Assume $\sqrt{13-\sqrt{160}} = \sqrt{x} - \sqrt{y}.$ (1)

Then, by Art. 253, $\sqrt{13+\sqrt{160}} = \sqrt{x} + \sqrt{y}.$ (2)

Multiplying (1) by (2), $\sqrt{169-160} = x - y.$

Or, $x - y = 3.$ (3)

Squaring (1), $13 - \sqrt{160} = x - 2\sqrt{xy} + y.$

Whence, by Art. 252, $x + y = 13.$ (4)

Adding (3) and (4), $2x = 16$, or $x = 8$.

Subtracting (3) from (4), $2y = 10$, or $y = 5$.

Substituting in (1),
$$\sqrt{13-\sqrt{160}} = \sqrt{8} - \sqrt{5}$$
$$= 2\sqrt{2} - \sqrt{5}, \text{ } Ans.$$

255. Examples like the above may often be solved by inspection by expressing the given quantity in the form of a perfect trinomial square (Art. 108), as follows:

Reduce the surd term so that its coefficient may be 2. Separate the rational term into two parts whose product is the quantity under the radical sign. Extract the square roots of these parts, and connect them by the sign of the surd term.

1. Extract the square root of $8 + \sqrt{48}$.

$$\sqrt{8+\sqrt{48}} = \sqrt{8+2\sqrt{12}}.$$

We then separate 8 into two parts whose product is 12. The parts are 6 and 2; hence,

$$\sqrt{8+2\sqrt{12}} = \sqrt{6+2\sqrt{6\times 2}+2}$$
$$= \sqrt{6} + \sqrt{2}, \text{ } Ans.$$

2. Extract the square root of $22-3\sqrt{32}$.

$$\sqrt{22-3\sqrt{32}}=\sqrt{22-\sqrt{288}}=\sqrt{22-2\sqrt{72}}.$$

We then separate 22 into two parts whose product is 72. The parts are 18 and 4; hence,

$$\sqrt{22-3\sqrt{32}}=\sqrt{18-2\sqrt{72}+4}$$
$$=\sqrt{18}-\sqrt{4}=3\sqrt{2}-2,\ \textit{Ans.}$$

EXAMPLES.

256. Extract the square roots of the following:

1. $12+2\sqrt{35}$. **6.** $8-\sqrt{60}$. **11.** $23+\sqrt{360}$.

2. $7-2\sqrt{12}$. **7.** $15+4\sqrt{14}$. **12.** $24-2\sqrt{63}$.

3. $9+2\sqrt{8}$. **8.** $12-\sqrt{108}$. **13.** $33+20\sqrt{2}$.

4. $9-4\sqrt{5}$. **9.** $20-5\sqrt{12}$. **14.** $47-6\sqrt{10}$.

5. $16+6\sqrt{7}$. **10.** $14+3\sqrt{20}$. **15.** $67-7\sqrt{72}$.

16. $2m-2\sqrt{m^2-n^2}$. **17.** $2a+x+2\sqrt{a^2+ax}$

SOLUTION OF EQUATIONS CONTAINING RADICALS

257. 1. Solve the equation $\sqrt{x^2-5}-x=-1$.

Transposing, $\sqrt{x^2-5}=x-1$.

Squaring both members, $x^2-5=x^2-2x+1$.

Transposing and uniting terms, $2x=6$.

$x=3$, *Ans.*

2. Solve the equation $\sqrt{2x-1}+\sqrt{2x+6}=7$.

Transposing $\sqrt{2x-1}$,

$$\sqrt{2x+6}=7-\sqrt{2x-1}.$$

Squaring, $\quad 2x+6=49-14\sqrt{2x-1}+2x-1.$

Transposing and uniting,

$$14\sqrt{2x-1}=42.$$

Or, $\quad \sqrt{2x-1}=3.$

Squaring, $\quad 2x-1=9.$

$$2x=10.$$

$$x=5, \textit{ Ans.}$$

RULE.

Transpose the terms of the equation so that a radical term may stand alone in one member; then raise both members to a power of the same degree as the radical.

If there are still radical terms remaining, repeat the operation.

Note. The equation should be simplified as much as possible before performing the involution.

EXAMPLES.

3. $\sqrt{5x-1}-2=1.$

4. $5=\sqrt[3]{2x}+3.$

5. $\sqrt[3]{4x+3}=3.$

6. $\sqrt{4x^2-19}-2x=-1.$

7. $\sqrt{x^2-3x+6}=2-x.$

8. $\sqrt[3]{x^3-6x^2}-x+2=0.$

9. $\sqrt{x}+\sqrt{x+5}=5.$

10. $\sqrt{x-32}+\sqrt{x}=16.$

11. $\sqrt{x-3}-\sqrt{x+12}=-3.$

12. $\sqrt{2x-7}+\sqrt{2x+9}=8.$

13. $\sqrt{3x+10}-\sqrt{3x+25}=-3.$

14. $\sqrt{(x-a)^2+2ab+b^2}=x-a+b.$

15. $\sqrt{x^2-3x+5}-\sqrt{x^2-5x-2}=1.$

16. $\sqrt{x}-\sqrt{x-3}=\dfrac{2}{\sqrt{x}}.$

17. $\sqrt{x-1}+\sqrt{x+4}=\sqrt{4x+5}.$

18. $\sqrt{x^2+4x+12}+\sqrt{x^2-12x-20}=8.$

19. $\dfrac{\sqrt{x}-3}{\sqrt{x}+7}=\dfrac{\sqrt{x}-4}{\sqrt{x}+1}.$

20. $\sqrt{3x}+\sqrt{3x+13}=\dfrac{91}{\sqrt{3x+13}}.$

21. $\sqrt{x+1}+\sqrt{x-2}-\sqrt{4x-3}=0.$

22. $\sqrt{x}+\sqrt{x+a}=\dfrac{2a}{\sqrt{x+a}}.$

23. $\sqrt{\{9+x\sqrt{x^2-3}\}}=x-3.$

24. $\dfrac{a}{\sqrt{a-x}}-\dfrac{x}{\sqrt{b-x}}=\sqrt{b-x}.$

25. $\sqrt{x+a}+\sqrt{x+b}=\sqrt{4x+a+3b}.$

26. $\sqrt{\{1+x\sqrt{x^2+16}\}}=x+1.$

27. $\sqrt{\{a^2-2ax+x^2\sqrt{3a-x}\}}=a-x$

XXI. QUADRATIC EQUATIONS.

258. A **Quadratic Equation,** or an equation of the *second degree* (Art. 167), is one in which the *square* is the highest power of the unknown quantity.

A **Pure Quadratic Equation** is one which contains only the square of the unknown quantity; as, $ax^2 = b$.

An **Affected Quadratic Equation** is one which contains both the square and the first power of the unknown quantity; as, $ax^2 + bx + c = 0$.

PURE QUADRATIC EQUATIONS.

259. A pure quadratic equation is solved by reducing it to the form $x^2 = a$, and then extracting the square roots of both members.

1. Solve the equation $3x^2 + 7 = \dfrac{5x^2}{4} + 35$.

Clearing of fractions, $12x^2 + 28 = 5x^2 + 140$.

Transposing and uniting, $7x^2 = 112$.

Or, $x^2 = 16$.

Taking the square root of both members,

$$x = \pm 4, \textit{ Ans.}$$

Note 1. The double sign is placed before the result because the square root of a number is either positive or negative (Art. 201).

2. Solve the equation $7x^2 - 5 = 5x^2 - 13$.

Transposing and uniting, $2x^2 = -8$.

Or, $x^2 = -4$.

Whence,
$$\begin{aligned} x &= \pm\sqrt{-4} \\ &= \pm 2\sqrt{-1}, \textit{ Ans.} \end{aligned}$$

Note 2. Since the square root of a negative quantity is imaginary (Art. 246), the values of x can only be indicated.

EXAMPLES.

Solve the following equations:

3. $4x^2 - 7 = 29.$

4. $5x^2 + 5 = 3x^2 + 55.$

5. $\dfrac{5}{6x^2} - \dfrac{7}{4x^2} = -\dfrac{33}{16}.$

6. $4 - \sqrt{3x^2 + 16} = 6.$

7. $ax^2 + b = c.$

8. $\dfrac{5}{4-x} = \dfrac{8}{3} - \dfrac{5}{4+x}.$

9. $2(x+3)(x-3) = (x+1)^2 - 2x.$

10. $(3x-2)(2x+5) + (5x+1)(4x-3) - 91 = 0.$

11. $\dfrac{x^2}{2} - 3 + \dfrac{5x^2}{12} = \dfrac{7}{24} - x^2 + \dfrac{335}{24}.$

12. $\dfrac{2x^2-5}{3} - \dfrac{3x^2+2}{7} - \dfrac{x^2-10}{6} = 0.$

13. $\dfrac{a}{x^2-b} = \dfrac{b}{x^2-a}.$

14. $\dfrac{4x^2-3}{2x^2-1} = \dfrac{2(9x^2+2)}{3(3x^2+2)}.$

15. $(2x-a)(x-b) + (2x+a)(x+b) = a^2 + b^2.$

16. $\dfrac{5x^2-1}{x^2-3} - \dfrac{3x^2+1}{x^2+2} - \dfrac{89}{(x^2-3)(x^2+2)} = 2.$

17. $x + \sqrt{x^2+3} = \dfrac{6}{\sqrt{x^2+3}}.$

18. $\dfrac{1}{1-\sqrt{1-x^2}} - \dfrac{1}{1+\sqrt{1-x^2}} = \dfrac{\sqrt{3}}{x^2}.$

AFFECTED QUADRATIC EQUATIONS.

260. An affected quadratic equation is solved by adding to both members such a quantity as will make the first member a perfect square; an operation which is termed *completing the square.*

FIRST METHOD OF COMPLETING THE SQUARE.

261. Every affected quadratic equation can be reduced to the form

$$x^2 + px = q;$$

where p and q represent any quantities whatever, positive or negative, integral or fractional.

Let it be required to solve the equation $x^2 + 3x = 4$.

In any trinomial square (Art. 108), the middle term is twice the product of the square roots of the first and third terms; hence the square root of the third term is equal to *the second term divided by twice the square root of the first.*

Therefore the *square root* of the quantity which must be added to $x^2 + 3x$ to make it a perfect square, is $\frac{3x}{2x}$, or $\frac{3}{2}$.

Adding to both members the square of $\frac{3}{2}$, or $\frac{9}{4}$, we have

$$x^2 + 3x + \frac{9}{4} = 4 + \frac{9}{4} = \frac{25}{4}.$$

Extracting the square root of both members,

$$x + \frac{3}{2} = \pm\frac{5}{2}.$$

Transposing $\frac{3}{2}$, $\qquad x = -\frac{3}{2} + \frac{5}{2}$, or $-\frac{3}{2} - \frac{5}{2}$.

Whence, $\qquad x = 1$ or -4, *Ans.*

262. From the above operation we derive the following rule:

Reduce the equation to the form $x^2 + px = q$.

Complete the square by adding to both members the square of half the coefficient of x.

Extract the square root of both members, and solve the simple equation thus formed.

1. Solve the equation $3x^2 - 8x = -4$.

Dividing by 3, $x^2 - \frac{8x}{3} = -\frac{4}{3}$,

which is in the form $x^2 + px = q$.

Adding to both members the square of $\frac{4}{3}$, or $\frac{16}{9}$,

$$x^2 - \frac{8x}{3} + \frac{16}{9} = -\frac{4}{3} + \frac{16}{9} = \frac{4}{9}.$$

Extracting the square root,

$$x - \frac{4}{3} = \pm\frac{2}{3}.$$

Whence, $x = \frac{4}{3} \pm \frac{2}{3} = 2$ or $\frac{2}{3}$, *Ans.*

Note. These values may be verified as follows:

Putting $x = 2$ in the given equation, $12 - 16 = -4$.

Putting $x = \frac{2}{3}$, $\frac{4}{3} - \frac{16}{3} = -4$.

If the coefficient of x^2 is negative, it is necessary to change the sign of each term.

2. Solve the equation $-3x^2 - 7x = \frac{10}{3}$.

Dividing by -3, $x^2 + \frac{7x}{3} = -\frac{10}{9}$.

Adding to both members the square of $\frac{7}{6}$, or $\frac{49}{36}$,

$$x^2 + \frac{7x}{3} + \frac{49}{36} = -\frac{10}{9} + \frac{49}{36} = \frac{9}{36}.$$

Extracting the square root,

$$x + \frac{7}{6} = \pm\frac{3}{6}.$$

Whence, $x = -\frac{7}{6} \pm \frac{3}{6} = -\frac{2}{3}$ or $-\frac{5}{3}$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $x^2+4x=5$.
4. $x^2-5x=-4$.
5. $x^2-7x=-12$.
6. $x^2+x=6$.
7. $3x^2-4x=4$.
8. $2x^2+5x=-2$.
9. $4x^2-8x+3=0$.
10. $4x^2-3=11x$.
11. $3-x-2x^2=0$.
12. $14+15x-9x^2=0$.

263. If the coefficient of x^2 is a perfect square, it is convenient to complete the square directly by the principle of Art. 261; that is, *by adding to both members the square of the quotient obtained by dividing the second term by twice the square root of the first.*

1. Solve the equation $9x^2-5x=4$.

The quotient of the second term divided by twice the square root of the first, is $\frac{5}{6}$. Adding the square of $\frac{5}{6}$ to both members,

$$9x^2-5x+\frac{25}{36}=4+\frac{25}{36}=\frac{169}{36}.$$

Extracting the square root,

$$3x-\frac{5}{6}=\pm\frac{13}{6}.$$

$$3x=\frac{5}{6}\pm\frac{13}{6}=3 \text{ or } -\frac{4}{3}.$$

Whence, $\quad x=1$ or $-\frac{4}{9}$, *Ans.*

Note. If the coefficient of x^2 is not a perfect square, it may be made so by multiplication.

Thus, in the equation $18x^2+5x=2$, the coefficient of x^2 may be made a perfect square by multiplying each term by 2.

If the coefficient of x^2 is negative, the sign of each term must be changed.

EXAMPLES.

Solve the following equations:

2. $4x^2+3x=10.$
3. $9x^2+2x=11.$
4. $25x^2-15x=-2.$
5. $4x^2-7x=-3.$
6. $2x^2+15x=-13.$
7. $8x^2+x-34=0.$
8. $11x+12-36x^2=0.$
9. $6x^2-5x=-1.$
10. $32x^2+20x-7=0.$
11. $48x^2-32x=3.$

SECOND METHOD OF COMPLETING THE SQUARE.

264. Every affected quadratic can be reduced to the form

$$ax^2+bx=c.$$

Multiplying each term by $4a$, we have

$$4a^2x^2+4abx=4ac.$$

Completing the square by adding to both members the square of b (Art. 263),

$$4a^2x^2+4abx+b^2=b^2+4ac.$$

Extracting the square root,

$$2ax+b=\pm\sqrt{b^2+4ac}.$$

Transposing, $$2ax=-b\pm\sqrt{b^2+4ac}.$$

Dividing by $2a$, $$x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

265. From the above operation we derive the following rule:

Reduce the equation to the form $ax^2+bx=c$.

Multiply both members by four times the coefficient of x^2, and add to each the square of the coefficient of x in the given equation.

Extract the square root of both members, and solve the simple equation thus formed.

Note. The advantage of this method over the preceding is in avoiding fractions in completing the square.

1. Solve the equation $2x^2 - 7x = -3$.

Multiplying both members by 4 times 2, or 8,

$$16x^2 - 56x = -24.$$

Adding to each member the square of 7,

$$16x^2 - 56x + 49 = -24 + 49 = 25.$$

Extracting the square root,

$$4x - 7 = \pm 5.$$

Transposing, $\quad 4x = 7 \pm 5 = 12 \text{ or } 2.$

Dividing by 4, $\quad x = 3 \text{ or } \frac{1}{2}$, *Ans.*

If the coefficient of x in the given equation is an *even* number, fractions may be avoided, and the rule modified, as follows:

Multiply both members by the coefficient of x^2, and add to each the square of half the coefficient of x in the given equation.

2. Solve the equation $7x^2 + 4x = 51$.

Multiplying both members by 7,

$$49x^2 + 28x = 357.$$

Adding to each member the square of 2,

$$49x^2 + 28x + 4 = 361.$$

Extracting the square root,

$$7x + 2 = \pm 19.$$

$$7x = -2 \pm 19 = 17 \text{ or } -21.$$

Whence, $\quad x = \frac{17}{7} \text{ or } -3$, *Ans.*

EXAMPLES.

Solve the following equations:

3. $2x^2+5x=3$.
4. $4x^2-x=3$.
5. $x^2-3x=18$.
6. $3x^2+4x=4$.
7. $8x^2+2x=3$.
8. $2x^2-7x=15$.
9. $7x^2-16x+4=0$.
10. $17x+20=-3x^2$.
11. $5x^2-3=14x$.
12. $2+x-6x^2=0$.
13. $8x^2+6x+1=0$.
14. $7x+3=6x^2$.
15. $15x^2-8x=-1$.
16. $41x-14-15x^2=0$.

MISCELLANEOUS EXAMPLES.

266. The following equations may be solved by either of the preceding methods; preference being given to the one best adapted to the example considered.

1. $\frac{x^2}{2}+\frac{x}{3}+\frac{1}{24}=0$.
2. $\frac{2}{x}+\frac{x}{2}=-\frac{5}{2}$.
3. $\frac{1}{2x}=\frac{7}{6x^2}-\frac{2}{3}$.
4. $\frac{2}{5}-\frac{5}{2x}=-\frac{15}{4x^2}$.
5. $(x+5)(x-5)-(11x+1)=0$.
6. $4x(18x-1)=(10x-1)^2$.
7. $(3x-5)^2-(x+2)^2=-5$.
8. $(x+3)^3-(x-1)^3=19$.
9. $(x-1)^2-(3x+8)^2-(2x+5)^2=0$.
10. $\frac{2x+3}{8+x}-\frac{2x+9}{3x+4}=0$.
11. $\frac{5}{x}-\frac{3x+1}{x^2}=\frac{1}{4}$.
12. $4x-\frac{14-x}{x+1}=14$.
13. $\frac{21}{5-x}-\frac{x}{7}=\frac{25}{7}$.

14. $\dfrac{3x^2}{x-7}-\dfrac{1-8x}{10}=\dfrac{x}{5}.$

15. $\dfrac{x}{x-1}-\dfrac{x-1}{x}=\dfrac{3}{2}.$

16. $\dfrac{x}{5-x}-\dfrac{5-x}{x}=\dfrac{15}{4}.$

17. $\dfrac{x+1}{x+2}-\dfrac{x+3}{x+4}=\dfrac{8}{3}.$

18. $\sqrt{20+x-x^2}=2x-10.$

19. $2\sqrt{x}+\dfrac{2}{\sqrt{x}}=5.$

20. $\dfrac{2x-1}{x}-\dfrac{3x}{3x-1}+\dfrac{1}{2}=0.$

21. $\dfrac{x^3-x^2+7}{x^2+3x-1}=x+\dfrac{11}{3}.$

22. $\dfrac{2x^3+3x-5}{3x^2+4x-1}=\dfrac{2x^2-x-1}{3x^2-2x+7}.$

23. $\dfrac{7}{x^2-4}-\dfrac{3}{x+2}=\dfrac{22}{5}.$

24. $\dfrac{1}{x^2-1}+\dfrac{1}{3}=\dfrac{1}{3(x-1)}+\dfrac{1}{x+1}.$

25. $\dfrac{x+3}{x+2}+\dfrac{x-3}{x-2}=\dfrac{2x-3}{x-1}.$

26. $\dfrac{12+5x}{12-5x}+\dfrac{2+x}{x}=\dfrac{1}{1-5x}.$

27. $\sqrt{4x-3}-\sqrt{x+1}=1.$

28. $2\sqrt{x}=\sqrt{x+5}+\dfrac{3}{\sqrt{x+5}}.$

29. $\dfrac{x+2}{x-1}=\dfrac{2x+16}{x+5}-\dfrac{x-2}{x+1}.$

30. $\sqrt{3x+1}+\sqrt{2x-1}=\sqrt{9x+4}.$

The same methods are applicable to the solution of literal quadratic equations.

31. Solve the equation $x^2 - 2mx = 2m + 1$.

Completing the square by adding m^2 to both members,

$$x^2 - 2mx + m^2 = m^2 + 2m + 1 = (m + 1)^2.$$

Extracting the square root,

$$x - m = \pm (m + 1).$$

Whence,
$$x = m + (m + 1), \text{ or } m - (m + 1)$$
$$= 2m + 1 \text{ or } -1, \textit{ Ans.}$$

32. Solve the equation $x^2 + ax - bx - ab = 0$.

The equation may be written,

$$x^2 + (a - b)x = ab.$$

Multiplying both members by 4 times the coefficient of x^2,

$$4x^2 + 4(a - b)x = 4ab.$$

Adding to each member the square of $a - b$,

$$4x^2 + 4(a - b)x + (a - b)^2 = (a - b)^2 + 4ab$$
$$= a^2 + 2ab + b^2.$$

Extracting the square root,

$$2x + (a - b) = \pm (a + b).$$

Whence,
$$2x = -(a - b) \pm (a + b).$$

Hence,
$$2x = -a + b + a + b = 2b,$$

or,
$$2x = -a + b - a - b = -2a.$$

Dividing by 2,
$$x = -a \text{ or } b, \textit{ Ans.}$$

Note. If several terms contain the same power of x, the coefficient of that power should be placed in a parenthesis, as shown in Ex. 32.

Solve the following equations:

33. $x^2 - 2ax = (b + a)(b - a)$.

34. $x^2 - ax + bx = ab$.

35. $x^2 - (a + 1)x = -a$.

36. $x^2 + 2(c+8)x = -32c.$

37. $x^2 - m^2(1-m)x = m^5.$

38. $acx^2 - bcx - adx = -bd.$

39. $(x+2p)^3 = (x+p)^3 + 37p^3.$

40. $6x^2 + 9ax + 2bx = -3ab.$

41. $\dfrac{2x(a-x)}{3a-2x} = \dfrac{a}{4}.$

42. $\dfrac{x^2}{x+1} = \dfrac{m^2}{m+1}.$

43. $x + \dfrac{1}{x} = \dfrac{a}{b} + \dfrac{b}{a}.$

44. $\dfrac{x+1}{\sqrt{x}} = \dfrac{a+1}{\sqrt{a}}.$

45. $\sqrt{(a+b)x - 4ab} = x - 2b.$

46. $\sqrt{x - 4ab} = \dfrac{(a+b)(a-b)}{\sqrt{x}}.$

47. $2\sqrt{x-m} + 3\sqrt{2x} = \dfrac{7m+5x}{\sqrt{x-m}}.$

48. $\dfrac{1}{a+\sqrt{a^2-x}} + \dfrac{1}{a-\sqrt{a^2-x}} = 1 + \dfrac{x}{a}.$

49. $\sqrt{x+a} + \sqrt{x+2a} = \sqrt{2x+3a}.$

50. $\dfrac{x^2+1}{x} = \dfrac{a+b}{c} + \dfrac{c}{a+b}.$

51. $\dfrac{1}{a+b+x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}.$

52. $x^2 + bx + cx = (a+c)(a-b).$

53. $abx^2 + \dfrac{3a^2x}{c} = \dfrac{6a^2 + ab - 2b^2}{c^2} - \dfrac{b^2x}{c}.$

54. $(3a^2 + b^2)(x^2 - x + 1) = (3b^2 + a^2)(x^2 + x + 1).$

SOLUTION OF QUADRATIC EQUATIONS BY A FORMULA.

267. It was shown in Art. 264 that if $ax^2+bx=c$, then

$$x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}. \qquad (1)$$

This result may be used as a formula for the solution of quadratic equations, as follows:

1. Solve the equation $3x^2+5x=12$.

In this case, $a=3$, $b=5$, $c=12$; substituting these values in (1),

$$x=\frac{-5\pm\sqrt{25+144}}{6}=\frac{-5\pm\sqrt{169}}{6}$$

$$=\frac{-5\pm 13}{6}=-3 \text{ or } \frac{4}{3}, \textit{ Ans.}$$

2. Solve the equation $110x^2-21x=-1$.

In this case, $a=110$, $b=-21$, $c=-1$; therefore,

$$x=\frac{21\pm\sqrt{441-440}}{220}=\frac{21\pm 1}{220}=\frac{1}{10} \text{ or } \frac{1}{11}, \textit{ Ans.}$$

Note. Particular attention must be paid to the *signs* of the coefficients in substituting.

EXAMPLES.

Solve the following equations:

3. $2x^2+5x=18$.	8. $5x^2-11x=-2$.
4. $3x^2-2x=5$.	9. $4x^2-8x-5=0$.
5. $x^2-7x=-10$.	10. $6x^2+25x+14=0$.
6. $5x^2+x=18$.	11. $30x-16=9x^2$.
7. $6x^2+7x=-1$.	12. $27+39x-10x^2=0$.

XXII. PROBLEMS.

INVOLVING QUADRATIC EQUATIONS.

268. **1.** A man sold a watch for \$21, and lost as much per cent as the watch cost him. Required the cost of the watch.

Let $x =$ the cost in dollars.

Then, $x =$ the loss per cent,

and $x \times \frac{x}{100} = \frac{x^2}{100} =$ the loss in dollars.

By the conditions, $\frac{x^2}{100} = x - 21.$

Solving this equation, $x = 70$ or 30.

That is, the cost of the watch was either \$70 or \$30; for each of these values satisfies the given conditions.

2. A farmer bought some sheep for \$72. If he had bought 6 more for the same money, they would have cost \$1 apiece less. How many did he buy?

Let $x =$ the number bought.

Then, $\frac{72}{x} =$ the price paid for one,

and $\frac{72}{x+6} =$ the price if there had been 6 more.

By the conditions, $\frac{72}{x} = \frac{72}{x+6} + 1.$

Solving, $x = 18$ or -24.

Only the positive value of x is admissible, as the negative value does not answer to the conditions of the problem. The number of sheep, therefore, was 18.

Note 1. In solving problems which involve quadratics, there will always be two values of the unknown quantity; but only those values should be retained as answers which satisfy the conditions of the problem.

Note 2. If, in the given problem, the words "6 more" had been changed to "6 *fewer*," and "$1 apiece less" to "$1 apiece *more*," we should have found the answer 24.

In many cases where the solution of a problem gives a negative result, the wording may be changed so as to form an analogous problem to which the absolute value of the negative result is an answer.

PROBLEMS.

3. I bought a lot of flour for $175; and the number of dollars per barrel was $\frac{1}{7}$ of the number of barrels. How many barrels were purchased, and at what price?

4. Separate the number 15 into two parts the sum of whose squares shall be 117.

5. Find two numbers whose product is 126, and quotient $3\frac{1}{2}$.

6. I have a rectangular field of corn containing 6250 hills. The number of hills in the length exceeds the number in the breadth by 75. How many hills are there in the length, and in the breadth?

7. Find two numbers whose difference is 9, and whose sum multiplied by the greater is 266.

8. The sum of the squares of two consecutive numbers is 113. What are the numbers?

9. A man cut two piles of wood, whose united contents were 26 cords, for $35.60. The labor on each cost as many dimes per cord as there were cords in the pile. Required the number of cords in each pile.

10. Find two numbers whose sum is 8, and the sum of whose cubes is 152.

11. Find three consecutive numbers such that twice the product of the first and third is equal to the square of the second increased by 62.

12. A grazier bought a certain number of oxen for $240. Having lost 3, he sold the remainder at $8 a head more than they cost him, and gained $59. How many did he buy?

13. A merchant bought a quantity of flour for $96. If he had bought 8 barrels more for the same money, he would have paid $2 less per barrel. How many barrels did he buy, and at what price?

14. Find two numbers, whose product is 78, such that if one be divided by the other the quotient is 2, and the remainder 1.

15. The plate of a rectangular looking-glass is 18 inches by 12. It is to be framed with a frame all parts of which are of the same width, and whose area is equal to that of the glass. Required the width of the frame.

16. A merchant sold a quantity of flour for $39, and gained as much per cent as the flour cost him. What was the cost of the flour?

17. A certain company agreed to build a vessel for $6300; but, two of their number having died, the rest had each to advance $200 more than they otherwise would have done. Of how many persons did the company consist at first?

18. Divide the number 24 into two parts, such that the sum of the fractions obtained by dividing 24 by them shall be $\frac{64}{15}$.

19. A detachment from an army was marching in regular column, with 6 men more in depth than in front. When the enemy came in sight, the front was increased by 870 men, and the whole was thus drawn up in 4 lines. Required the number of men.

20. A merchant sold goods for $16, and lost as much per cent as the goods cost him. What was the cost of the goods?

21. A certain farm is a rectangle, whose length is twice its breadth. If it should be enlarged 20 rods in length, and 24 rods in breadth, its area would be doubled. Of how many acres does the farm consist?

22. A square court-yard has a gravel-walk around it. The side of the court lacks one yard of being 6 times the breadth of the walk, and the number of square yards in the walk exceeds the number of yards in the perimeter of the court by 340. Find the area of the court and the width of the walk.

23. A merchant bought 54 bushels of wheat, and a certain quantity of barley. For the former he gave half as many dimes per bushel as there were bushels of barley, and for the latter 40 cents a bushel less. He sold the mixture at $1 per bushel, and lost $57.60 by the operation. Required the quantity of barley, and its price per bushel.

24. A certain number consists of two digits, the left-hand digit being twice the right-hand. If the digits are inverted, the product of the number thus formed, increased by 11, and the original number, is 4956. Find the number.

25. A cistern can be filled by two pipes running together in 2 hours 55 minutes. The larger pipe by itself will fill it sooner than the smaller by 2 hours. What time will each pipe separately take to fill it?

26. A and B gained by trade $1800. A's money was in the firm 12 months, and he received for his principal and gain $2600. B's money, which was $3000, was in the firm 16 months. How much money did A put into the firm?

27. My gross income is $1000. After deducting a percentage for income tax, and then a percentage, less by one than that of the income tax, from the remainder, the income is reduced to $912. Find the rate per cent of the income tax.

28. A man travelled 102 miles. If he had gone 3 miles more an hour, he would have performed the journey in $5\frac{2}{3}$ hours less time. How many miles an hour did he go?

29. The number of square inches in the surface of a cubical block exceeds the number of inches in the sum of its edges by 210. What is its volume?

30. A man has two square lots of unequal size, containing together 15,025 square feet. If the lots were contiguous, it would require 530 feet of fence to embrace them in a single enclosure of six sides. Required the area of each lot.

31. A set out from C towards D at the rate of 3 miles an hour. After he had gone 28 miles, B set out from D towards C, and went every hour $\frac{1}{19}$ of the entire distance; and after he had travelled as many hours as he went miles in an hour, he met A. Required the distance from C to D.

32. A courier proceeds from P to Q in 14 hours. A second courier starts at the same time from a place 10 miles behind P, and arrives at Q at the same time as the first courier. The second courier finds that he takes half an hour less than the first to accomplish 20 miles. Find the distance from P to Q.

33. A person bought a number of $20 mining-shares when they were at a certain rate per cent discount for $1500; and afterwards, when they were at the same rate per cent premium, sold them all but 60 for $1000. How many did he buy, and what did he give for each of them?

XXIII. EQUATIONS IN THE QUADRATIC FORM.

269. An equation is in the *quadratic form* when it is expressed in three terms, two of which contain the unknown quantity; and of these two, *one has an exponent twice as great as the other;* as,

$$x^6 - 6x^3 = 16;$$

$$x^3 + x^{\frac{3}{2}} = 72;$$

$$(x^2-1)^2 + 3(x^2-1) = 18; \text{ etc.}$$

270. The rules for the solution of quadratics are applicable to equations having the same form.

1. Solve the equation $x^6 - 6x^3 = 16$.

Completing the square,

$$x^6 - 6x^3 + 9 = 16 + 9 = 25.$$

Extracting the square root,

$$x^3 - 3 = \pm 5.$$

Whence, $\quad x^3 = 3 \pm 5 = 8 \text{ or } -2.$

Extracting the cube root, $x = 2$ or $-\sqrt[3]{2}$, *Ans.*

Note. There are also four imaginary roots, which may be obtained by the method explained in Art. 282.

2. Solve the equation $2x + 3\sqrt{x} = 27$.

Since $\sqrt{x}$ is the same as $x^{\frac{1}{2}}$, this is in the quadratic form.

Multiplying by 8, and adding 3^2 or 9 to both members,

$$16x + 24\sqrt{x} + 9 = 216 + 9 = 225.$$

Extracting the square root,

$$4\sqrt{x} + 3 = \pm 15.$$

Or, $\quad 4\sqrt{x} = -3 \pm 15 = 12 \text{ or } -18.$

Whence, $$\sqrt{x} = 3 \text{ or } -\frac{9}{2}.$$

Squaring, $$x = 9 \text{ or } \frac{81}{4}, \textit{ Ans.}$$

3. Solve the equation $16x^{-\frac{3}{2}} - 22x^{-\frac{3}{4}} = 3$.

Multiplying by 16, and adding 11^2 to both members,

$$16^2x^{-\frac{3}{2}} - 16 \times 22x^{-\frac{3}{4}} + 121 = 48 + 121 = 169.$$

Extracting the square root,

$$16x^{-\frac{3}{4}} - 11 = \pm 13.$$

Or, $$16x^{-\frac{3}{4}} = 11 \pm 13 = -2 \text{ or } 24.$$

Whence, $$x^{-\frac{3}{4}} = -\frac{1}{8} \text{ or } \frac{3}{2}.$$

Extracting the cube root,

$$x^{-\frac{1}{4}} = -\frac{1}{2} \text{ or } \left(\frac{3}{2}\right)^{\frac{1}{3}}.$$

Raising to the fourth power,

$$x^{-1} = \frac{1}{16} \text{ or } \left(\frac{3}{2}\right)^{\frac{4}{3}}.$$

Inverting both members, $x = 16 \text{ or } \left(\frac{2}{3}\right)^{\frac{4}{3}}$, *Ans.*

Note. In solving equations of the form $x^{\frac{p}{q}} = a$, first extract the root corresponding to the numerator, and afterwards raise to the power corresponding to the denominator. Particular attention should be paid to the algebraic signs; see Arts 192 and 201.

EXAMPLES.

Solve the following equations:

4. $x^4 - 25x^2 = -144$.

5. $x^6 + 20x^3 - 69 = 0$.

6. $x^{10} + 31x^5 - 32 = 0$.

7. $x^{-4} - 9x^{-2} = -20$.

8. $81x^2 + \frac{1}{x^2} = 82$.

9. $8x^6 - 216 = 37x^3$.

10. $(3x^2-2)^2-11(3x^2-2)+10=0.$

11. $(x^3-5)^2=241-29x^3.$

12. $x^3-x^{\frac{3}{2}}=56.$

13. $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756.$

14. $2x^{\frac{2}{n}}+3x^{\frac{4}{n}}-56=0.$

15. $3x^{\frac{5}{3}}+x^{\frac{5}{6}}=3104.$

16. $3x^{\frac{3}{2}}+26x^{\frac{3}{4}}=-16.$

17. $2x^{-5}+61x^{-\frac{5}{2}}-96=0.$

18. $4x-15=17\sqrt{x}.$

19. $\dfrac{\sqrt{4x}+2}{4+\sqrt{x}}=\dfrac{4}{\sqrt{x}}-1.$

20. $3x^{\frac{4}{3}}-\dfrac{5x^{\frac{8}{3}}}{2}=-592.$

21. $8x^{-\frac{6}{5}}-15x^{-\frac{3}{5}}-2=0.$

271. An equation may be solved with reference to an expression, by regarding it as a single quantity.

1. Solve the equation $(x-5)^3-3(x-5)^{\frac{3}{2}}=40.$

Regarding $x-5$ as a single quantity, we complete the square in the usual way. Multiplying by 4, and adding 9 to both members,

$$4(x-5)^3-12(x-5)^{\frac{3}{2}}+9=160+9=169.$$

Extracting the square root,

$$2(x-5)^{\frac{3}{2}}-3=\pm 13.$$

Or, $\quad 2(x-5)^{\frac{3}{2}}=3\pm 13=16 \text{ or } -10.$

Whence, $\quad (x-5)^{\frac{3}{2}}=8 \text{ or } -5.$

Extracting the cube root,

$$(x-5)^{\frac{1}{2}}=2 \text{ or } -\sqrt[3]{5}.$$

Squaring, $\quad x-5=4 \text{ or } \sqrt[3]{25}.$

Transposing, $\quad x=9 \text{ or } 5+\sqrt[3]{25},$ *Ans.*

An equation of the fourth degree may sometimes be solved by expressing it in the quadratic form.

2. Solve the equation $x^4+12x^3+34x^2-12x-35=0$.

We may write the equation as follows:

$$(x^4+12x^3+36x^2)-2x^2-12x=35.$$

Or, $$(x^2+6x)^2-2(x^2+6x)=35.$$

Completing the square,

$$(x^2+6x)^2-2(x^2+6x)+1=36.$$

Extracting the square root, $(x^2+6x)-1=\pm 6$.

Whence, $x^2+6x=1\pm 6=\ 7$ or -5.

Completing the square, $x^2+6x+9=16$ or 4.

Extracting the square root,

$$x+3=\pm 4 \text{ or } \pm 2.$$

Whence, $$x=-3\pm 4 \text{ or } -3\pm 2$$
$$=1,\ -7,\ -1, \text{ or } -5, \textit{ Ans.}$$

Note. In solving equations like the above, the first step is to form a perfect square with the x^4 and x^3 terms, and a portion of the x^2 term. By Art. 261, the third term of the square is the square of the quotient obtained by dividing the x^3 term by twice the square root of the x^4 term.

3. Solve the equation $2x^2+\sqrt{2x^2+1}=11$.

Adding 1 to both members,

$$(2x^2+1)+\sqrt{2x^2+1}=12.$$

Completing the square,

$$4(2x^2+1)+4\sqrt{2x^2+1}+1=48+1=49.$$

Extracting the square root,

$$2\sqrt{2x^2+1}+1=\pm 7.$$

Or, $$2\sqrt{2x^2+1}=-1\pm 7=6 \text{ or } -8.$$

Whence, $$\sqrt{2x^2+1}=3 \text{ or } -4.$$

Squaring, $$2x^2 + 1 = 9 \text{ or } 16.$$

$$2x^2 = 8 \text{ or } 15.$$

$$x^2 = 4 \text{ or } \frac{15}{2}.$$

Extracting the square root, $x = \pm 2$ or $\pm \frac{1}{2}\sqrt{30}$, *Ans.*

Note. In solving equations of this form, add such quantities to both members that the expression without the radical in the first member may be the same as that within, or some multiple of it.

EXAMPLES.

Solve the following equations:

4. $(x^2 - 5x)^2 - 8(x^2 - 5x) = 84.$

5. $x^4 + 10x^3 + 17x^2 - 40x - 84 = 0.$

6. $x^2 - 10x - 2\sqrt{x^2 - 10x + 18} + 15 = 0.$

7. $x^2 + 5 + \sqrt{x^2 + 5} = 12.$

8. $2x^2 + 3x - 5\sqrt{2x^2 + 3x + 9} = -3.$

9. $x^4 + 2x^3 - 25x^2 - 26x + 120 = 0.$

10. $x^4 - 6x^3 - 29x^2 + 114x = 80.$

11. $x^2 - 6x + 5\sqrt{x^2 - 6x + 20} = 46.$

12. $\sqrt{x + 10} - \sqrt[4]{x + 10} = 2.$

13. $4x^2 + 6\sqrt{4x^2 + 12x - 2} = -3 - 12x.$

14. $(x^3 + 16)^{\frac{2}{3}} - 3(x^3 + 16)^{\frac{1}{3}} + 2 = 0.$

15. $4(x - 1)^{\frac{4}{3}} - 5(x - 1)^{\frac{2}{3}} + 1 = 0.$

16. $x^4 + 14x^3 + 47x^2 - 14x - 48 = 0.$

17. $3(x^2 + 5x) - 2\sqrt{x^2 + 5x + 1} = 2.$

18. $(x - a)^{\frac{5}{3}} + 2\sqrt{b}(x - a)^{\frac{5}{6}} - 3b = 0.$

XXIV. SIMULTANEOUS EQUATIONS.

INVOLVING QUADRATICS.

272. The degree of an equation containing more than one unknown quantity is determined by the greatest sum of the exponents of the unknown quantities in any term. Thus,

$2x + 3xy = 4$ is an equation of the second degree.

$x^2 - ay^2z = ab^3$ is an equation of the third degree.

Note. This definition assumes that the equation has been cleared of fractions, and freed from radical signs and fractional and negative exponents.

273. Two equations of the second degree with two unknown quantities will generally produce, by elimination, an equation of the fourth degree with one unknown quantity. The rules for quadratics are, therefore, not sufficient to solve all simultaneous equations of the second degree.

In several cases, however, the solution may be effected by the ordinary rules.

Case I.

274. *When one equation is of the first degree.*

Equations of this kind may always be solved by finding the value of one of the unknown quantities in terms of the other from the simple equation, and substituting the result in the other equation.

1. Solve the equations $\begin{cases} 2x^2 - xy = 6y. & (1) \\ x + 2y = 7. & (2) \end{cases}$

From (2), $\quad 2y = 7 - x$, or $y = \dfrac{7-x}{2}$. (3)

Substituting in (1), $2x^2 - x\left(\dfrac{7-x}{2}\right) = 6\left(\dfrac{7-x}{2}\right)$.

Clearing of fractions, $4x^2-7x+x^2=42-6x.$

Or, $5x^2-x=42.$

Solving this equation, $x=3$ or $-\frac{14}{5}.$

Substituting in (3), $y=\frac{7-3}{2}$ or $\frac{7+\frac{14}{5}}{2}$

$=2$ or $\frac{49}{10}.$

Ans. $x=3,\ y=2$; or, $x=-\frac{14}{5},\ y=\frac{49}{10}.$

EXAMPLES.

Solve the following equations:

2. $\begin{cases} 2x^2-3y^2=-10. \\ 3x+\ \ y=1. \end{cases}$

3. $\begin{cases} x+y=-1. \\ xy=-56. \end{cases}$

4. $\begin{cases} x-y=\ \ 3. \\ x^2+y^2=117. \end{cases}$

5. $\begin{cases} 10x+y=3xy. \\ x-y=-2. \end{cases}$

6. $\begin{cases} x^3-y^3=-37. \\ x-y=-\ \ 1. \end{cases}$

7. $\begin{cases} x-y=5. \\ xy=-6. \end{cases}$

8. $\begin{cases} x+y=\ 3. \\ x^2+y^2=29. \end{cases}$

9. $\begin{cases} \frac{x}{2}+\frac{y}{3}=4. \\ \frac{2}{x}+\frac{3}{y}=1. \end{cases}$

10. $\begin{cases} x^3+y^3=152. \\ x+y=\ \ 2. \end{cases}$

11. $\begin{cases} 3x^2-2xy=15. \\ 2x+3y=12. \end{cases}$

12. $\begin{cases} 8x^3-y^3=-7. \\ 2x-y=-1. \end{cases}$

13. $\begin{cases} x^2+3xy-y^2=23. \\ x+2y=7. \end{cases}$

14. $\begin{cases} \frac{x}{y}+\frac{y}{x}=\frac{5}{2}. \\ 3x-2y=-4. \end{cases}$

Case II.

275. *When the equations are symmetrical with respect to x and y.*

Note 1. An equation is symmetrical with respect to two quantities when they can be interchanged without destroying the equality.

Thus, $x^2 - xy + y^2 = 3$ is symmetrical, for on interchanging x and y it becomes $y^2 - yx + x^2 = 3$, which is equivalent to the first equation. But $x - y = 1$ is not symmetrical, for on interchanging x and y it becomes $y - x = 1$, which is a different equation.

In solving equations by the symmetrical method, they must be combined in such a way as to give the values of the sum and difference of the unknown quantities.

1. Solve the equations $\begin{cases} x + y = 2. & (1) \\ xy = -15. & (2) \end{cases}$

Squaring (1), $x^2 + 2xy + y^2 = 4.$ (3)

Multiplying (2) by 4, $4xy = -60.$ (4)

Subtracting (4) from (3), $x^2 - 2xy + y^2 = 64.$

Extracting the square root, $x - y = \pm 8.$ (5)

Adding (1) and (5), $2x = 2 \pm 8 = 10$ or -6.

Whence, $x = 5$ or -3.

Subtracting (5) from (1), $2y = 2 \mp 8 = -6$ or 10.

Whence, $y = -3$ or 5.

Ans. $x = 5,\ y = -3$; or, $x = -3,\ y = 5$.

Note 2. The signs $\pm$ and $\mp$ before two quantities signify that when the first quantity is $+$, the second is $-$; and when the first is $-$, the second is $+$. Thus, in the above solution, when $2x = 2 + 8$, $2y = 2 - 8$; and when $2x = 2 - 8$, $2y = 2 + 8$. That is, when $x = 5$, $y = -3$; and when $x = -3$, $y = 5$.

In the operation, the sign $\pm$ is changed to $\mp$ whenever $+$ would be changed to $-$.

Note 3. The above equations may also be solved as in Case I.; but the symmetrical method is shorter, and more elegant.

2. Solve the equations $\begin{cases} x^2+y^2=50. & (1) \\ x-y=8. & (2) \end{cases}$

Squaring (2), $x^2-2xy+y^2=64.$ (3)

Subtracting (3) from (1), $2xy=-14.$ (4)

Adding (1) and (4), $x^2+2xy+y^2=36.$

Whence, $x+y=\pm 6.$ (5)

Adding (2) and (5), $2x=8\pm 6=14$ or $2.$

Whence, $x=7$ or $1.$

Subtracting (2) from (5), $2y=-8\pm 6=-2$ or $-14.$

Whence, $y=-1$ or $-7.$

Ans. $x=7,\ y=-1$; or, $x=1,\ y=-7.$

Note 4. The symmetrical method may often be used in cases like the above, where the equations are symmetrical except in the signs of the terms.

3. Solve the equations $\begin{cases} x^3+y^3=133. & (1) \\ x^2-xy+y^2=19. & (2) \end{cases}$

Dividing (1) by (2), $x+y=7.$ (3)

Squaring, $x^2+2xy+y^2=49.$ (4)

Subtracting (4) from (2), $-3xy=-30.$

Or, $-xy=-10.$ (5)

Adding (2) and (5), $x^2-2xy+y^2=9.$

Whence, $x-y=\pm 3.$ (6)

Adding (3) and (6), $2x=7\pm 3=10$ or $4.$

Whence, $x=5$ or $2.$

Subtracting (6) from (3), $2y=7\mp 3=4$ or $10.$

Whence, $y=2$ or $5.$

Ans. $x=5,\ y=2$; or, $x=2,\ y=5.$

EXAMPLES.

Solve the following equations:

4. $\begin{cases} x+y=1. \\ xy=-6. \end{cases}$

5. $\begin{cases} x-y=6. \\ x^2+y^2=90. \end{cases}$

6. $\begin{cases} x-y=-10. \\ xy=-21. \end{cases}$

7. $\begin{cases} x^3+y^3=-19. \\ x^2-xy+y^2=19. \end{cases}$

8. $\begin{cases} x^2+y^2=25. \\ xy=12. \end{cases}$

9. $\begin{cases} x+y=-4. \\ x^2+y^2=58. \end{cases}$

10. $\begin{cases} x^3-y^3=98. \\ x-y=2. \end{cases}$

11. $\begin{cases} x^3+y^3=9. \\ x+y=3. \end{cases}$

12. $\begin{cases} x-y=\frac{1}{2}. \\ xy=60. \end{cases}$

13. $\begin{cases} x^2+y^2=85. \\ xy=42. \end{cases}$

14. $\begin{cases} x^3-y^3=-316. \\ x-y=-4. \end{cases}$

15. $\begin{cases} x^2+y^2=193. \\ x+y=-5. \end{cases}$

16. $\begin{cases} x+y=12. \\ xy=-45. \end{cases}$

17. $\begin{cases} x^3-y^3=-65. \\ x^2+xy+y^2=13. \end{cases}$

18. $\begin{cases} x^2+y^2=157. \\ x-y=-5. \end{cases}$

19. $\begin{cases} x^3+y^3=-386. \\ x+y=-2. \end{cases}$

CASE III.

276. *When each equation is of the second degree, and homogeneous* (Art. 35).

Note. Certain examples, in which the equations are of the second degree and homogeneous, may be solved by the method of Case II. The method of Case III. should be used only when the equations can be solved in no other way.

1. Solve the equations $\begin{cases} x^2 - 2xy = 5. \\ x^2 + y^2 = 29. \end{cases}$

Putting $y = vx$ in the given equations, we have

$$x^2 - 2vx^2 = 5; \text{ or, } x^2 = \frac{5}{1-2v}. \qquad (1)$$

$$x^2 + v^2x^2 = 29; \text{ or, } x^2 = \frac{29}{1+v^2}.$$

Equating the values of x^2, $\dfrac{5}{1-2v} = \dfrac{29}{1+v^2}$.

$$5 + 5v^2 = 29 - 58v.$$

$$5v^2 + 58v = 24.$$

Solving this equation, $v = \dfrac{2}{5}$ or -12.

Substituting these values in (1), $x^2 = \dfrac{5}{1-\frac{4}{5}}$ or $\dfrac{5}{1+24}$

$$= 25 \text{ or } \frac{1}{5}.$$

Whence, $x = \pm 5$ or $\pm \dfrac{1}{\sqrt{5}}$.

Substituting the values of v and x in the equation $y = vx$,

$$y = \frac{2}{5}(\pm 5) \text{ or } -12\left(\pm \frac{1}{\sqrt{5}}\right) = \pm 2 \text{ or } \mp \frac{12}{\sqrt{5}}.$$

Ans. $x = \pm 5, \quad y = \pm 2;$

or, $x = \pm \dfrac{1}{5}\sqrt{5}$, $y = \mp \dfrac{12}{5}\sqrt{5}$.

Note. In finding y from the equation $y = vx$, care must be taken to multiply each pair of values of x by the *corresponding value* of v.

EXAMPLES.

Solve the following equations:

2. $\begin{cases} x^2 - xy = 35. \\ xy + y^2 = 18. \end{cases}$

3. $\begin{cases} 2x^2 + xy = 15. \\ x^2 - y^2 = 8. \end{cases}$

4. $\begin{cases} x^2 + xy - y^2 = -11. \\ x^2 + y^2 = 13. \end{cases}$

5. $\begin{cases} x^2 + xy + 4y^2 = 6. \\ 3x^2 + 8y^2 = 14. \end{cases}$

6. $\begin{cases} x^2 + xy = 12. \\ xy - y^2 = 2. \end{cases}$

7. $\begin{cases} 2y^2 - 4xy + 3x^2 = 17. \\ y^2 - x^2 = 16. \end{cases}$

8. $\begin{cases} 2x^2 - 2xy - y^2 = 3. \\ x^2 + 3xy + y^2 = 11. \end{cases}$

9. $\begin{cases} 6x^2 - 5xy + 2y^2 = 12. \\ 3x^2 + 2xy - 3y^2 = -3. \end{cases}$

10. $\begin{cases} x^2 + xy - y^2 = 1. \\ x^2 - xy + 2y^2 = 8. \end{cases}$

11. $\begin{cases} 4xy - x^2 = 5. \\ 13x^2 - 31xy + 16y^2 = 2\frac{1}{2}. \end{cases}$

MISCELLANEOUS EXAMPLES.

277. No general rules can be given for the solution of examples which do not come under the cases just considered. Various artifices are employed, familiarity with which can only be obtained by experience.

1. Solve the equations $\begin{cases} x^3 - y^3 = 19. & (1) \\ x^2y - xy^2 = 6. & (2) \end{cases}$

Multiplying (2) by 3, $3x^2y - 3xy^2 = 18.$ (3)

Subtracting (3) from (1),

$$x^3 - 3x^2y + 3xy^2 - y^3 = 1.$$

Extracting the cube root, $x - y = 1.$ (4)

Dividing (2) by (4), $xy = 6.$ (5)

Equations (4) and (5) may now be solved by the method of Case II. We shall find $x = 3$ or -2, and $y = 2$ or -3.

Ans. $x = 3,\ y = 2$; or, $x = -2,\ y = -3$.

2. Solve the equations $\begin{cases} x^3 + y^3 = 9xy. \\ x + y = 6. \end{cases}$

Putting $x = u + v$ and $y = u - v$, we have

$$(u+v)^3 + (u-v)^3 = 9(u+v)(u-v). \quad (1)$$

$$(u+v) + (u-v) = 6. \quad (2)$$

Reducing (2), $2u = 6$, or $u = 3$.

Reducing (1), $2u^3 + 6uv^2 = 9(u^2 - v^2)$.

Substituting the value of u,

$$54 + 18v^2 = 9(9 - v^2).$$

Whence, $v^2 = 1$, or $v = \pm 1$.

Therefore, $x = u + v = 3 \pm 1 = 4$ or 2,

$y = u - v = 3 \mp 1 = 2$ or 4.

Ans. $x = 4$, $y = 2$; or, $x = 2$, $y = 4$.

Note. The artifice of substituting $u + v$ and $u - v$ for x and y is advantageous in any case where the given equations are *symmetrical*.

3. Solve the equations

$$\begin{cases} x^2 + y^2 + 2x + 2y = 23. & (1) \\ xy = 6. & (2) \end{cases}$$

Multiplying (2) by 2, $2xy = 12$. (3)

Adding (1) and (3),

$$x^2 + 2xy + y^2 + 2x + 2y = 35.$$

Or, $(x + y)^2 + 2(x + y) = 35$.

Completing the square,

$$(x + y)^2 + 2(x + y) + 1 = 36.$$

Whence, $(x + y) + 1 = \pm 6$,

$x + y = -1 \pm 6 = 5$ or -7. (4)

Squaring (4), $x^2 + 2xy + y^2 = 25$ or 49. (5)

Multiplying (2) by 4, $4xy = 24$. (6)

Subtracting (6) from (5),

$$x^2 - 2xy + y^2 = 1 \text{ or } 25.$$

Whence, $x-y=\pm 1$ or ± 5. (7)

Adding (4) and (7), $2x=5\pm 1$ or -7 ± 5,

$x=3, 2, -1$, or -6.

Subtracting (7) from (4), $2y=5\mp 1$, or -7 ∓ 5,

$y=2, 3, -6$, or -1.

Ans. $x=3, y=2$; $x=2, y=3$;

$x=-1, y=-6$; or, $x=-6, y=-1$.

4. Solve the equations $\begin{cases} x^4+y^4= \ \ 97. \\ x\ +y\ =-1. \end{cases}$

Putting $x=u+v$ and $y=u-v$, we have,

$$(u+v)^4+(u-v)^4= \ \ 97. \qquad (1)$$

$$(u+v)\ +(u-v)\ =-1. \qquad (2)$$

Reducing (2), $2u=-1$, or $u=-\frac{1}{2}$.

Reducing (1), $2u^4+12u^2v^2+2v^4= \ \ 97.$

Substituting the value of u,

$$\frac{1}{8}+3v^2+2v^4= \ \ 97.$$

Solving this equation, $v^2=\frac{25}{4}$, or $-\frac{31}{4}$.

Whence, $v=\pm\frac{5}{2}$, or $\pm\frac{\sqrt{-31}}{2}$.

Therefore, $x=u+v=-\frac{1}{2}\pm\frac{5}{2}$ or $-\frac{1}{2}\pm\frac{\sqrt{-31}}{2}$

$$=2 \text{ or } -3 \text{ or } \frac{-1\pm\sqrt{-31}}{2};$$

and $y=u-v=-\frac{1}{2}\mp\frac{5}{2}$ or $-\frac{1}{2}\mp\frac{\sqrt{-31}}{2}$

$$=-3 \text{ or } 2 \text{ or } \frac{-1\mp\sqrt{-31}}{2}.$$

EXAMPLES.

Solve the following equations:

5. $\begin{cases} xy - 2x = 5. \\ xy + 3y = -2. \end{cases}$

6. $\begin{cases} x + y = 9. \\ \sqrt[3]{x} + \sqrt[3]{y} = 3. \end{cases}$

7. $\begin{cases} 4x^2 - 3y^2 = -11. \\ 11x^2 + 5y^2 = 301. \end{cases}$

8. $\begin{cases} x^3 + y^3 = 35. \\ x^2y + xy^2 = 30. \end{cases}$

9. $\begin{cases} \dfrac{x}{y} + \dfrac{y}{x} = \dfrac{29}{10}. \\ 3x - 2y = 4. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 5m^2. \\ x - y = m. \end{cases}$

11. $\begin{cases} x^2 + y^2 + x + y = 18. \\ xy = 6. \end{cases}$

12. $\begin{cases} x^2 - 2xy = 16. \\ 2xy + y^2 = -3. \end{cases}$

13. $\begin{cases} x^3 + y^3 = 18xy. \\ x + y = 12. \end{cases}$

14. $\begin{cases} x^2 + 3xy = -14. \\ xy + 4y^2 = 30. \end{cases}$

15. $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 11. \\ \dfrac{1}{xy} = 18. \end{cases}$

16. $\begin{cases} x - y = a - b. \\ xy = 2a^2 + 2ab. \end{cases}$

17. $\begin{cases} x^2 + y^2 = 9 - x. \\ x^2 - y^2 = 6. \end{cases}$

18. $\begin{cases} \dfrac{1}{x^2} + \dfrac{1}{y^2} = 65. \\ \dfrac{1}{x} - \dfrac{1}{y} = 11. \end{cases}$

19. $\begin{cases} x^2 + y^2 - x - y = 18. \\ xy + x + y = 19. \end{cases}$

20.* $\begin{cases} x^2 + xy + y^2 = 7. \\ x^4 + x^2y^2 + y^4 = 133. \end{cases}$

21. $\begin{cases} x^2y + xy^2 = 6. \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{2}{3}. \end{cases}$

22. $\begin{cases} x^4 + y^4 = 17. \\ x - y = 3. \end{cases}$

23. $\begin{cases} x^3 - y^3 = 7a^3. \\ x - y = a. \end{cases}$

* Divide the second equation by the first.

24. $\begin{cases} x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y = 19. \\ x^2 + xy + y^2 = 133. \end{cases}$

25. $\begin{cases} x + 2y = 3a + b. \\ xy + y^2 = 2a(a + b). \end{cases}$

26. $\begin{cases} x^2y + xy^2 = 30. \\ x^4y^2 + x^2y^4 = 468. \end{cases}$

27. $\begin{cases} x^3 - y^3 = 6a^2b + 2b^3. \\ xy(x - y) = 2a^2b - 2b^3. \end{cases}$

28. $\begin{cases} x^2 + 3x + y = 73 - 2xy. \\ y^2 + 3y + x = 44. \end{cases}$

29. $\begin{cases} \dfrac{x}{y} + \dfrac{4\sqrt{x}}{\sqrt{y}} = \dfrac{33}{4}. \\ x - y = 5. \end{cases}$

30. $\begin{cases} x^2 - xy + y^2 = 19. \\ 2x^2 - y^2 = -17. \end{cases}$

31. $\begin{cases} x + y = 3(a - b). \\ xy = 2a^2 - 5ab + 2b^2. \end{cases}$

32. $\begin{cases} x^5 + y^5 = 33. \\ x + y = 3. \end{cases}$

33. $\begin{cases} xy^2 + y = 1. \\ x^2y^4 + y^2 = 5. \end{cases}$

34. $\begin{cases} x + z = 7. \\ 2y - 3z = -5. \\ x^2 + y^2 - z^2 = 11. \end{cases}$

35. $\begin{cases} 2x^2 - 7xy - 2y^2 = 5. \\ 3xy - x^2 + 6y^2 = 44. \end{cases}$

36. $\begin{cases} \dfrac{x - y}{x + y} - \dfrac{x + y}{x - y} = \dfrac{3}{2}. \\ 2x - y = 7. \end{cases}$

37. $\begin{cases} x^2 + y^2 = 7 + xy. \\ x^3 + y^3 = 6xy - 1. \end{cases}$

PROBLEMS.

278. Note. In the following problems, as in those of Chap. XXII., only those answers are to be retained which satisfy the conditions of the problem.

1. The sum of the squares of two numbers is 106, and the difference of their squares is $\frac{7}{2}$ the square of their difference. Find the numbers.

2. What two numbers are those whose difference multiplied by the less produces 42, and by their sum, 133?

3. The sum of the areas of two square fields is 1300 square rods, and it requires 200 rods of fence to enclose both. What are the areas of the fields?

4. The difference of the squares of two numbers is 7, and the product of their squares is 144. Find the numbers.

5. If the length of a rectangular field were increased by 2 rods, and its breadth by 3 rods, its area would be 108 square rods; and if its length were diminished by 2 rods, and its breadth by 3 rods, its area would be 24 square rods. Find the length and breadth of the field.

6. The sum of the cubes of two numbers is 407, and the sum of their squares exceeds their product by 37. Find the numbers.

7. A man bought 6 ducks and 2 turkeys for \$15. He bought four more ducks for \$14 than turkeys for \$9. What was the price of each?

8. Find a number of two figures, such that if its digits are inverted, the sum of the number thus formed, and the original number, is 33, and their product is 252.

9. The sum of the terms of a fraction is 8. If 1 is added to each term, the product of the resulting fraction and the original fraction is $\frac{2}{5}$. Required the fraction.

10. A rectangular garden is surrounded by a walk 7 feet wide; the area of the garden is 15,000 square feet, and of the walk 3696 square feet. Find the length and breadth of the garden.

11. A rectangular field contains 160 square rods. If its length be increased by 4 rods, and its breadth by 3 rods, its area is increased by 100 square rods. Find the length and breadth of the field.

12. A man rows down stream 12 miles in 4 hours less time than it takes him to return. Should he row at twice his ordinary rate, his rate down stream would be 10 miles an hour. Find his rate in still water, and the rate of the stream.

13. A and B bought a farm of 104 acres, for which they paid $320 each. On dividing the land, A said to B, "If you will let me have my portion in the situation which I shall choose, you shall have so much more land than I, that mine shall cost $3 an acre more than yours." B accepted the proposal. How much did each have, and at what price per acre?

14. If the product of two numbers be added to their sum, the result is 47; and the sum of their squares exceeds their sum by 62. Find the numbers.

Note. Let the numbers be represented by $x+y$ and $x-y$.

15. The sum of two numbers is 7, and the sum of their fourth powers is 641. Required the numbers.

16. The fore-wheel of a carriage makes 15 more revolutions than the hind-wheel in going 180 yards; but if the circumference of each wheel were increased by 3 feet, the fore-wheel would make only 9 more revolutions than the hind-wheel in the same distance. Find the circumference of each wheel.

17. A man has $1300, which he divides into two portions, and loans at different rates of interest, so that the two portions produce equal returns. If the first portion had been loaned at the second rate, it would have produced $36; and if the second portion had been loaned at the first rate, it would have produced $49. Required the rates of interest.

18. Cloth, when wetted, shrinks $\frac{1}{8}$ in its length and $\frac{1}{16}$ in its width. If the surface of a piece of cloth is diminished by $5\frac{3}{4}$ square yards, and the length of the four sides by $4\frac{1}{4}$ yards, what were the length and width of the cloth originally?

XXV. THEORY OF QUADRATIC EQUATIONS.

279. Denoting the roots of the equation $x^2+px=q$ by r_1 and r_2, we have (Art. 267),

$$r_1=\frac{-p+\sqrt{p^2+4q}}{2}, \text{ and } r_2=\frac{-p-\sqrt{p^2+4q}}{2}.$$

Adding these values,

$$r_1+r_2=\frac{-2p}{2}=-p.$$

Multiplying them together,

$$r_1r_2=\frac{p^2-(p^2+4q)}{4} \text{ (Art. 95) } =\frac{-4q}{4}=-q.$$

That is, *if a quadratic equation be reduced to the form* $x^2+px=q$, *the algebraic sum of the roots is equal to the coefficient of x with its sign changed, and the product of the roots is equal to the second member, with its sign changed.*

Example. Required the sum and product of the roots of the equation $2x^2-7x-15=0$.

The equation may be written in the form

$$x^2-\frac{7x}{2}=\frac{15}{2}.$$

Whence, the sum of the roots is $\frac{7}{2}$, and their product is $-\frac{15}{2}$.

280. The principles of Art. 279 may be used to form a quadratic equation which shall have any required roots.

For, denoting the roots of the equation $x^2+px-q=0$ by r_1 and r_2, we have, by the preceding article,

$$p=-(r_1+r_2), \text{ and } -q=r_1r_2.$$

We may therefore write the equation in the form

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0,$$

or,
$$x^2 - r_1x - r_2x + r_1r_2 = 0.$$

That is (Art. 105), $(x - r_1)(x - r_2) = 0.$

Hence, to form an equation which shall have any required roots,

Subtract each of the roots from x, and place the product of the resulting expressions equal to zero.

Example. Form the equation whose roots are 4 and $-\frac{7}{4}$.

By the rule, $$(x-4)\left(x+\frac{7}{4}\right)=0.$$

Multiplying by 4, $$(x-4)(4x+7)=0.$$

Or, $$4x^2 - 9x - 28 = 0, \textit{ Ans.}$$

EXAMPLES.

281. Find by inspection the sum and product of the roots of:

1. $x^2 + 5x + 2 = 0.$
2. $x^2 - 7x + 11 = 0.$
3. $x^2 + 6x - 1 = 0.$
4. $2x^2 - 3x - 2 = 0.$
5. $8x^2 - x + 4 = 0.$
6. $6x - 4x^2 + 3 = 0.$
7. $7 - 12x - 14x^2 = 0.$
8. $4x^2 - 4ax + a^2 - b^2 = 0.$

Form the equations whose roots are:

9. 4, 5.
10. 1, -3.
11. 3, $-\frac{3}{5}$.
12. 7, $-\frac{19}{3}$.
13. $\frac{2}{3}, \frac{3}{4}$.
14. $-\frac{8}{3}, \frac{4}{7}$.
15. $-\frac{5}{3}, -\frac{7}{2}$.
16. $-\frac{17}{3}$, 0.
17. $a-b$, $a+2b$.
18. $m(1+m)$, $m(1-m)$.
19. $2+\sqrt{3}$, $2-\sqrt{3}$.
20. $\frac{m+\sqrt{n}}{2}, \frac{m-\sqrt{n}}{2}$.

282. By Art. 280, the equation $x^2+px-q=0$ may be written in the form $(x-r_1)(x-r_2)=0$, where r_1 and r_2 are its roots.

It will be observed that the roots may be obtained by *placing the factors of the first member separately equal to zero, and solving the simple equations thus formed.*

This principle is often used in solving equations:

1. Solve the equation $(2x-3)(3x+5)=0$.

Placing the factors separately equal to zero,

$$2x-3=0, \text{ or } x=\frac{3}{2};$$

and $$3x+5=0, \text{ or } x=-\frac{5}{3}.$$

Ans. $x=\frac{3}{2}$, or $-\frac{5}{3}$.

2. Solve the equation $x^3-5x^2-24x=0$.

Factoring the first member, $x(x-8)(x+3)=0$.

Therefore, $$x=0;$$

$$x-8=0, \text{ or } x=8;$$

and $$x+3=0, \text{ or } x=-3.$$

Ans. $x=0$, 8, or -3.

3. Solve the equation $x^3+4x^2-x-4=0$.

Factoring the first member (Art. 105),

$$(x+4)(x^2-1)=0.$$

Therefore, $$x+4=0, \text{ or } x=-4;$$

and $$x^2-1=0, \text{ or } x=\pm 1.$$

Ans. $x=-4$ or ± 1.

4. Solve the equation $x^3 - 1 = 0$.

Factoring the first member,

$$(x-1)(x^2+x+1)=0.$$

Therefore, $x - 1 = 0$, or $x = 1$;

and $x^2 + x + 1 = 0$, or $x = \dfrac{-1 \pm \sqrt{-3}}{2}$ (Art. 267).

Ans. $x = 1$ or $\dfrac{-1 \pm \sqrt{-3}}{2}$.

EXAMPLES.

Solve the following equations:

5. $\left(x - \frac{3}{5}\right)\left(x + \frac{7}{2}\right) = 0$.

6. $2x^2 - x = 0$.

7. $(ax + b)(bx - a) = 0$.

8. $(x^2 - 4)(x^2 - 9) = 0$.

9. $(x-2)(x^2+9x+20)=0$.

10. $2x^3 - 18x = 0$.

11. $(3x + 1)(4x^2 - 25) = 0$.

12. $3x^3 + 12x^2 = 0$.

13. $(x^2 - a^2)(x^2 - ax - b) = 0$.

14. $24x^3 - 2x^2 - 12x = 0$.

15. $x(2x + 5)(3x - 7)(4x + 1) = 0$.

16. $(x^2 - 5x + 6)(x^2 + 7x + 12)(x^2 - 3x - 4) = 0$.

17. $x^3 + 1 = 0$.

18. $8x^3 = 27$.

19. $x^6 - 1 = 0$.

20. $x^3 + x^2 + x + 1 = 0$.

21. $x^3 - x^2 - 9x + 9 = 0$.

22. $2x^3 + 3x^2 - 2x - 3 = 0$.

23. $x^3 - ax^2 + a^2x - a^3 = 0$.

24. $12x^3+8x^2-27x-18=0$.

Note. The above examples are illustrations of the important principle that, the degree of an equation indicates the number of its roots; thus, an equation of the third degree has three roots; of the fourth degree, four roots; etc.

It should be observed that the roots are not necessarily *unequal*; thus, the equation $x^2 - 2x + 1 = 0$ may be written $(x-1)(x-1) = 0$, and therefore the two roots are 1 and 1.

FACTORING.

283. A *quadratic expression* is a trinomial expression of the form $ax^2 + bx + c$.

Any such expression may be resolved into two simple factors by the artifice of completing the square (Art. 260), in connection with Art. 111.

EXAMPLES.

1. Factor $6x^2 + 7x - 3$.

$$6x^2 + 7x - 3 = 6\left(x^2 + \frac{7x}{6} - \frac{1}{2}\right).$$

By Art. 262, the expression in the parenthesis will become a perfect square if the third term is $\left(\frac{7}{12}\right)^2$; hence,

$$\begin{aligned} 6x^2 + 7x - 3 &= 6\left[x^2 + \frac{7x}{6} + \left(\frac{7}{12}\right)^2 - \left(\frac{7}{12}\right)^2 - \frac{1}{2}\right] \\ &= 6\left[\left(x + \frac{7}{12}\right)^2 - \frac{121}{144}\right] \\ &= 6\left(x + \frac{7}{12} + \frac{11}{12}\right)\left(x + \frac{7}{12} - \frac{11}{12}\right) \quad \text{(Art. 111)} \\ &= 6\left(x + \frac{3}{2}\right)\left(x - \frac{1}{3}\right) \\ &= 2\left(x + \frac{3}{2}\right) \cdot 3\left(x - \frac{1}{3}\right) \\ &= (2x + 3)(3x - 1), \ \textit{Ans.} \end{aligned}$$

2. Factor $4x^2 - 20x + 19$.

$$\begin{aligned} 4x^2 - 20x + 19 &= 4x^2 - 20x + 25 - 25 + 19 \\ &= (2x - 5)^2 - 6 \\ &= (2x - 5 + \sqrt{6})(2x - 5 - \sqrt{6}), \ \textit{Ans.} \end{aligned}$$

Note. If the x^2 term is *negative*, the entire expression should be enclosed in a parenthesis preceded by a $-$ sign.

3. Factor $102+11x-x^2$.

$$102+11x-x^2=-(x^2-11x-102)$$
$$=-\left[x^2-11x+\left(\frac{11}{2}\right)^2-\left(\frac{11}{2}\right)^2-102\right]$$
$$=-\left[\left(x-\frac{11}{2}\right)^2-\frac{529}{4}\right]$$
$$=-\left(x-\frac{11}{2}+\frac{23}{2}\right)\left(x-\frac{11}{2}-\frac{23}{2}\right)$$
$$=(x+6)\cdot(-1)(x-17)$$
$$=(6+x)(17-x),\ \textit{Ans.}$$

4. Factor $x^2-xy-2y^2-5x+y+6$.

$$x^2-xy-5x-2y^2+y+6$$
$$=x^2-x(y+5)+\left(\frac{y+5}{2}\right)^2-\left(\frac{y+5}{2}\right)^2-2y^2+y+6$$
$$=\left(x-\frac{y+5}{2}\right)^2-\frac{y^2+10y+25}{4}-\frac{8y^2-4y-24}{4}$$
$$=\left(x-\frac{y+5}{2}\right)^2-\frac{9y^2+6y+1}{4}$$
$$=\left(x-\frac{y+5}{2}+\frac{3y+1}{2}\right)\left(x-\frac{y+5}{2}-\frac{3y+1}{2}\right)\quad\text{(Art. 111)}$$
$$=(x+y-2)(x-2y-3),\ \textit{Ans.}$$

Factor the following:

5. $x^2-4x-60$.

6. $x^2+13x+40$.

7. $x^2-11x+18$.

8. $2x^2-7x-15$.

9. $4x^2-15x+9$.

10. $5x^2+36x+7$.

11. $39-10x-x^2$.

12. $2+x-6x^2$.

13. x^2+4x+1.

14. $9x^2-6x-4$.

15. $8x^2 - 18x + 9$.

16. $6 - x - 2x^2$.

17. $5 + 4x - 12x^2$.

18. $9x^2 - 12x + 1$.

19. $5 - 18x - 8x^2$.

20. $10x^2 - 23x + 6$.

21. $16x^2 + 34x + 15$.

22. $1 - 8x - x^2$.

23. $15 + 26x - 24x^2$.

24. $25x^2 - 20x - 2$.

25. $6x^2 - 13ax - 15a^2$.

26. $20x^2 + 41mx + 20m^2$.

27. $12x^2 + 7xy - 10y^2$.

28. $21x^2 - 58mnx + 21m^2n^2$.

29. $x^2 + xy - 6y^2 + x + 13y - 6$.

30. $x^2 + 3xy + 2y^2 + 3x + 4y + 2$.

31. $6 - 5y + x - 6y^2 + 5xy - x^2$.

32. $x^2 - 5xy + 6y^2 - 5xz + 14yz + 4z^2$.

33. $2x^2 - xy - y^2 + 3x + 3y - 2$.

34. $3a^2 + 4ab + b^2 + 5a - b - 12$.

Certain expressions of the *fourth* degree may be resolved into two quadratic factors by the artifice of completing the square.

35. Factor $a^4 + a^2b^2 + b^4$.

By Art. 108, the expression will become a perfect square if the middle term is $2a^2b^2$. Hence,

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= (a^4 + 2a^2b^2 + b^4) - a^2b^2 \\ &= (a^2 + b^2)^2 - a^2b^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab) \qquad \text{(Art. 111)} \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2), \textit{ Ans.} \end{aligned}$$

36. Factor $9x^4 - 39x^2 + 25$.

$$\begin{aligned} 9x^4 - 39x^2 + 25 &= (9x^4 - 30x^2 + 25) - 9x^2 \\ &= (3x^2 - 5)^2 - 9x^2 \\ &= (3x^2 - 5 + 3x)(3x^2 - 5 - 3x) \\ &= (3x^2 + 3x - 5)(3x^2 - 3x - 5), \textit{ Ans.} \end{aligned}$$

37. Factor x^4+1.

$$x^4+1=(x^4+2x^2+1)-2x^2$$
$$=(x^2+1)^2-(x\sqrt{2})^2$$
$$=(x^2+x\sqrt{2}+1)(x^2-x\sqrt{2}+1),\ \textit{Ans.}$$

Factor the following:

38. x^4+x^2+1.

39. x^4-7x^2+1.

40. $4a^4-8a^2b^2+b^4$.

41. $m^4-14m^2n^2+n^4$.

42. $1-13b^2+4b^4$.

43. $x^4-12x^2y^2+4y^4$.

44. $4a^4+8a^2+9$.

45. $4m^4-24m^2+25$.

46. $a^4-5a^2x^2+x^4$.

47. x^4+81.

48. $4a^4+15a^2b^2+16b^4$.

49. $16x^4-49m^2x^2+9m^4$.

50. $9x^4-6x^2+4$.

51. $9a^4+14a^2m^2+25m^4$.

52. $4-32n^2+49n^4$.

53. $16x^4-49x^2y^2+25y^4$.

284. The equation $x^4+1=0$ may be solved, as in Art. 282, by placing the factors of the first member (Ex. 37, Art. 283) separately equal to zero; thus,

$$x^2+x\sqrt{2}+1=0;\ \text{whence } x=\frac{-\sqrt{2}\pm\sqrt{-2}}{2};$$

and $$x^2-x\sqrt{2}+1=0;\ \text{whence } x=\frac{\sqrt{2}\pm\sqrt{-2}}{2}.$$

Therefore, $$x=\frac{\pm\sqrt{2}\pm\sqrt{-2}}{2}.$$

EXAMPLES.

Solve the following equations:

1. $x^4+16=0$.

2. $x^4-6x^2+1=0$.

3. $x^4-x^2+1=0$.

4. $x^4+a^4=0$.

5. $x^4-8x^2+4=0$.

6. $x^4-\frac{3x^2}{2}+1=0$.

DISCUSSION OF THE GENERAL EQUATION.

285. The roots of the equation $x^2 + px = q$ are

$$r_1 = \frac{-p + \sqrt{p^2 + 4q}}{2} \text{ and } r_2 = \frac{-p - \sqrt{p^2 + 4q}}{2}.$$

We will now discuss these values for different values of p and q.

I. *Suppose q positive.*

Since p^2 is essentially positive (Art. 192), the quantity under the radical sign is positive and greater than p^2.

Therefore the value of the radical is greater than p.

Hence r_1 is positive and r_2 is negative.

If p is positive, r_2 is numerically greater than r_1; that is, the negative root is numerically the greater.

If p is zero, the roots are numerically equal.

If p is negative, r_1 is numerically greater than r_2; that is, the positive root is numerically the greater.

II. *Suppose $q = 0$.*

The quantity under the radical sign is now equal to p^2, so that the value of the radical is p.

If p is positive, $r_1 = 0$, and r_2 is negative.

If p is negative, r_1 is positive, and $r_2 = 0$.

III. *Suppose q negative, and $4q$ numerically $< p^2$.*

The quantity under the radical sign is now positive and less than p^2.

Therefore the value of the radical is less than p.

If p is positive, both roots are negative.

If p is negative, both roots are positive.

IV. *Suppose q negative, and $4q$ numerically $= p^2$.*

The quantity under the radical sign is now equal to zero.

Therefore the roots are equal; being negative if p is positive, and positive if p is negative.

V. *Suppose q negative, and $4q$ numerically $> p^2$.*

The quantity under the radical sign is now negative; hence, by Art. 201, both roots are imaginary.

The roots are both *rational* or both *irrational* according as $p^2 + 4q$ *is* or *is not* a perfect square.

EXAMPLES.

1. Determine by inspection the nature of the roots of the equation $2x^2 - 5x - 18 = 0$.

The equation may be written $x^2 - \frac{5x}{2} = 9$.

Since q is positive and p negative, the roots are one positive and the other negative; and the positive root is numerically the greater.

In this case, $p^2 + 4q = \frac{25}{4} + 36 = \frac{169}{4}$; a perfect square.

Hence the roots are both rational.

Determine by inspection the nature of the roots of the following:

2. $x^2 + 2x - 15 = 0$.
3. $x^2 + 5x + 6 = 0$.
4. $x^2 - 10x = -25$.
5. $3x^2 - 5x + 4 = 0$.
6. $6x^2 - 7x - 5 = 0$.
7. $9x^2 + 30x = -25$.
8. $9x^2 + 8 = 18x$.
9. $10 - 3x - 18x^2 = 0$.

XXVI. INEQUALITIES.

286. An **Inequality** is a statement that one of two quantities is greater or less than the other; as,

$$a > b, \text{ or } m < n.$$

The terms *greater* and *less* are here taken in the algebraic sense; that is, of any two quantities a and b, a is the *greater* when $a - b$ is *positive*, and the *less* when $a - b$ is *negative*.

287. The expression on the left of the sign of inequal'' is called the *First Member*, and that on the right the *Second Member*, of the inequality.

288. Two inequalities are said to *subsist in the same sense* when the first member is the greater or the less in each.

Thus,

$$a > b, \text{ and } c > d; \text{ or } m < n, \text{ and } p < q,$$

are inequalities which subsist in the same sense.

289. Two inequalities are said to *subsist in a contrary sense* when the first member is the greater in one, and the less in ''e other.

Thus, $$a > b, \text{ and } c < d$$

are inequalities which subsist in a contrary sense.

290. *An inequality will continue in the same sense after the same quantity has been added to, or subtracted from, both members.*

For consider the inequality $a > b$.

Then by Art. 286, $a - b$ is positive.

Therefore each of the quantities

$$(a + c) - (b + c), \text{ and } (a - c) - (b - c)$$

is positive, since each is equal to $a - b$.

Whence by Art. 286,

$$a + c > b + c, \text{ and } a - c > b - c.$$

It follows from the above that *a term may be transposed from one member of an inequality to the other by changing its sign.*

291. *If the signs of all the terms of an inequality are changed, the sign of inequality must be reversed.*

For consider the inequality

$$a - b > c - d.$$

Transposing each term (Art. 290), we have

$$d - c > b - a.$$

That is, $$b - a < d - c.$$

292. *An inequality will continue in the same sense after both members have been multiplied or divided by the same positive quantity.*

For consider the inequality $a > b$.

By Art. 286, $a - b$ is positive.

Hence, if m is positive, each of the quantities

$$m(a - b), \text{ and } \frac{a - b}{m},$$

or, $$ma - mb, \text{ and } \frac{a}{m} - \frac{b}{m},$$

is positive.

Therefore, $$ma > mb, \text{ and } \frac{a}{m} > \frac{b}{m}.$$

293. *If both members of an inequality are multiplied or divided by the same negative quantity, the sign of inequality must be reversed.*

For multiplying or dividing by a negative quantity changes the signs of all the terms, and hence the sign of inequality must be reversed (Art. 291).

294. ***If any number of inequalities, subsisting in the same sense, are added member to member, the resulting inequality will also subsist in the same sense.***

For consider the inequalities

$$a > b,\ a' > b',\ a'' > b'', \dots.$$

Then each of the quantities $a - b$, $a' - b'$, $a'' - b''$, ..., is positive.

Therefore their sum

$$a - b + a' - b' + a'' - b'' + \cdots,$$

or, $$a + a' + a'' + \cdots - (b + b' + b'' + \cdots)$$

is positive.

Whence, $a + a' + a'' + \cdots > b + b' + b'' + \cdots.$

Note. If two inequalities, subsisting in the same sense, are *subtracted* member from member, the resulting inequality will not necessarily subsist in the same sense.

Thus, if $a > b$ and $a' > b'$, then $a - b$ and $a' - b'$ are positive.

But $a - b - (a' - b')$, or its equal, $a - a' - (b - b')$, may be either positive, negative, or zero; and hence it does not necessarily follow that $a - a' > b - b'$.

EXAMPLES.

295. 1. Find the limit of x in the inequality

$$7x - \frac{23}{3} < \frac{2x}{3} + 5.$$

Clearing of fractions (Art. 292), we have

$$21x - 23 < 2x + 15.$$

Transposing (Art. 290), and uniting terms,

$$19x < 38.$$

Whence by Art. 292, $x < 2$, *Ans.*

2. Find the limits of x and y in the following:

$$\begin{cases} 3x + 2y > 37 & (1) \\ 2x + 3y = 33 & (2) \end{cases}$$

Multiplying (1) by 3, and (2) by 2,

$$9x + 6y > 111$$
$$4x + 6y = 66$$

Subtracting, $\quad 5x > 45$

Whence, $\quad x > 9.$

Multiplying (1) by 2, and (2) by 3,

$$6x + 4y > 74$$
$$6x + 9y = 99$$

Subtracting, $\quad -5y > -25$

Whence by Art. 293, $\quad y < 5.$

Therefore, $\quad x > 9$, and $y < 5$, *Ans.*

Find the limits of x in the following:

3. $(6x+1)^2 - 105 < (4x-3)(9x+4).$

4. $(2x+3)(3x-1) > (2x+7)(3x-2) + 1.$

5. $(x+1)(x+2)(x-3) > (x-1)(x-4)(x+5).$

6. $3ax + 14ab > 6a^2 + 7bx$, if $3a - 7b$ is negative.

7. $\dfrac{x-a}{b} < \dfrac{x-b}{a}$, if a and b are positive and $a > b$.

Find the limits of x and y in the following:

8. $\begin{cases} 5x + 7y > 38. \\ x - y = -2. \end{cases}$ **9.** $\begin{cases} 2x + 3y < 57. \\ 3x + 7y = 93. \end{cases}$

10. Find the limits of x when

$$2x - 9 > 21 - 4x, \text{ and } 3x - 11 > 5x - 41.$$

11. A certain positive whole number, plus 23, is less than 6 times the number, minus 12; and 9 times the number, minus 54, is less than twice the number, plus 9. What is the number?

12. A teacher being asked the number of his pupils, replied that 29 was less than twice their number, diminished by 7; and that 5 times their number, diminished by 5, was less than twice their number, increased by 55. Required the number of his pupils.

13. A shepherd has a number of sheep such that twice the number, diminished by 45, exceeds 79, diminished by twice the number; and 5 times the number, increased by 1, is less than 3 times the number, increased by 69. How many sheep has he?

14. Prove that if a and b are positive,

$$\frac{a}{b}+\frac{b}{a}>2.$$

Since the square of any quantity is positive,

$$(a-b)^2>0.$$

That is, $$a^2-2ab+b^2>0,$$

or, $$a^2+b^2>2ab.$$

Dividing each term of the inequality by ab (Art. 292), we have

$$\frac{a}{b}+\frac{b}{a}>2.$$

15. Prove that for any value of x, $x^2-3x+4>1\frac{3}{4}$.

16. Prove that for any values of a and b,

$$(2a+b)(2a-b)>2b(6a-5b).$$

17. Prove that for any values of a, b, and c,

$$a^2+b^2+c^2>2(ab+bc-ca).$$

18. Prove that $(a^2-b^2)(c^2-d^2)<(ac-bd)^2$.

19. Prove that if $a^2+b^2=1$ and $c^2+d^2=1$, then

$$ab+cd<1.$$

XXVII. THE THEORY OF LIMITS.

INTERPRETATION OF THE FORMS $\frac{a}{0}$, $\frac{a}{\infty}$, AND $\frac{0}{0}$.

Note. The symbol ∞ is called *Infinity*.

296. A *variable quantity*, or simply a *variable*, is a quantity which may assume, under the conditions imposed upon it, an indefinitely great number of different values.

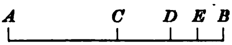
A *constant* is a quantity which remains unchanged throughout the same discussion.

297. A *limit* of a variable is a constant quantity, the difference between which and the variable may be made less than any assigned quantity however small, without ever becoming zero.

In other words, a limit of a variable is a fixed quantity to which the variable approaches indefinitely near, but never actually reaches.

The variable is said to *approach indefinitely* to its limit.

298. Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C, half-way between A and B; then to D, half-way between C and B; then to E, half-way between D and B; and so on indefinitely.

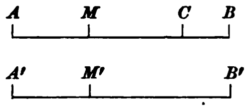


In this case the distance between the moving point and B can be made less than any assigned quantity however small, but cannot be made equal to zero.

Hence the distance from A to the moving point is a variable which approaches indefinitely the constant value AB as a limit, without ever reaching it.

Again, the distance from the moving point to B is a variable which approaches the limit 0.

299. The Theorem of Limits. *If two variables are always equal, and each approaches a limit, the two limits are equal.*



Let AM and $A'M'$ be two equal variables which approach the limits AB and $A'B'$, respectively.

If possible, suppose $AB > A'B'$, and lay off $AC = A'B'$.

Then the variable AM may assume values between AC and AB, while the variable $A'M'$ is restricted to values less than AC; which is contrary to the hypothesis that the variables should always be equal.

Hence AB cannot be $> A'B'$, and in like manner it may be proved that AB cannot be $< A'B'$; therefore $AB = A'B'$.

INTERPRETATION OF $\frac{a}{0}$.

300. Consider the series of fractions

$$\frac{a}{3},\ \frac{a}{.3},\ \frac{a}{.03},\ \frac{a}{.003},\ \cdots,$$

where each denominator is one-tenth of the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made less than any assigned quantity however small, and the value of the fraction may be made greater than any assigned quantity however great.

In other words,

If the numerator of a fraction remains constant, while the denominator approaches the limit 0, *the value of the fraction increases without limit.*

It is customary to express this principle as follows:

$$\frac{a}{0} = \infty.$$

INTERPRETATION OF $\frac{a}{\infty}$.

301. Consider the series of fractions

$$\frac{a}{3},\ \frac{a}{30},\ \frac{a}{300},\ \frac{a}{3000},\ \cdots,$$

where each denominator is ten times the preceding denominator.

It is evident that, by sufficiently continuing the series, the denominator may be made greater than any assigned quantity however great, while the value of the fraction may be made less than any assigned quantity however small.

In other words,

If the numerator of a fraction remains constant, while the denominator increases without limit, the value of the fraction approaches the limit 0.

It is customary to express this principle as follows:

$$\frac{a}{\infty} = 0.$$

302. The student must understand clearly that no *absolute meaning* can be attached to such results as

$$\frac{a}{0} = \infty, \text{ or } \frac{a}{\infty} = 0;$$

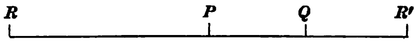
for there can be no such thing as division unless the divisor and quotient are *finite quantities.*

If such forms occur in mathematical investigations, they must be interpreted as indicated in Arts. 300 and 301. (Compare the Note to Art. 405.)

THE PROBLEM OF THE COURIERS.

303. The discussion of the following problem will serve to further illustrate the form $\frac{a}{0}$, besides furnishing an interpretation of the form $\frac{0}{0}$.

Two couriers, A and B, are travelling along the same road in the same direction, RR', at the rates of m and n miles an hour respectively. If at any time, say 12 o'clock, A is at P, and B is a miles from him at Q, at what time and at what point are they together?



Let $\quad x =$ the required time in hours after 12 o'clock,

and $\quad y =$ the distance travelled by A in the time x, or the distance in miles from P to the point of meeting.

Then $y - a =$ the distance travelled by B in the time x, or the distance in miles from Q to the point of meeting.

Since the distance equals the rate multiplied by the time,

$$\begin{cases} y = mx, \\ y - a = nx. \end{cases}$$

Solving these equations, we obtain

$$x = \frac{a}{m-n},$$

$$y = \frac{am}{m-n}.$$

We will now discuss these values under different hypotheses.

1. $m > n$.

In this case the values of x and y are *positive*.

Hence the couriers are together at some time *after* 12 o'clock, and at some point to the *right* of P.

This corresponds with the supposition made; for if m is greater than n, A is travelling faster than B, and it is evident that he will eventually overtake him at some point in advance of their positions at 12 o'clock.

2. $m < n$.

In this case the values of x and y are *negative*.

Hence the couriers are together at some time *before* 12 o'clock, and at some point to the *left* of P. (Compare Art. 44.)

This corresponds with the hypothesis; for if m is less than n, A is travelling more slowly than B, and they must have been together before 12 o'clock, and before they could have advanced as far as P.

3. $m = n$, or $m - n = 0$.

In this case the values of x and y take the forms

$$\frac{a}{0}, \text{ and } \frac{am}{0},$$

respectively.

As $m - n$ approaches the limit 0, the values of x and y increase without limit (Art. 300); hence, if $m = n$, no finite values can be assigned to x and y, and the problem is impossible.

This interpretation corresponds with the supposition made; for if m is equal to n, the couriers are a miles apart at 12 o'clock, and are travelling at the same rate; and it is evident that they never could have been, and never will be together.

Thus, *an infinite result indicates that the problem is impossible.*

4. $a = 0$, and $m > n$ or $m < n$.

In this case we have $x = 0$ and $y = 0$.

Hence the couriers are together at 12 o'clock, at the point P.

This corresponds with the hypothesis; for if $a = 0$, and m and n are unequal, the couriers are together at 12 o'clock, and are travelling at unequal rates; hence they never could have been together before that time, and they never will be together afterwards.

5. $a = 0$, and $m = n$.

In this case both x and y take the form $\frac{0}{0}$.

According to the supposition made, the couriers are together at 12 o'clock, and are travelling at the same rate.

Therefore they always must have been, and always will be together.

There is in this case no single answer nor finite number of answers to the problem; for any value of x whatever, together with the corresponding value of y, will satisfy the given conditions.

Hence, *a result* $\frac{0}{0}$ *indicates that the problem is indeterminate;* that is, the number of solutions is indefinitely great.

XXVIII. RATIO AND PROPORTION.

304. **Ratio** is the relation with respect to magnitude which one quantity bears to another of the same kind, and is expressed by writing the first quantity as the numerator and the second as the denominator of a fraction.

Thus the ratio of a to b is $\frac{a}{b}$; and it is also expressed $a:b$.

305. A **Proportion** is an equality of ratios.

Thus, if the ratio of a to b is equal to the ratio of c to d, they form a proportion, which may be written in either of the forms:

$$a:b=c:d, \quad \frac{a}{b}=\frac{c}{d}, \quad \text{or} \quad a:b::c:d.$$

306. The first term of a ratio is called the *antecedent*, and the second term the *consequent*.

Thus in the ratio $a:b$, a is the antecedent, and b is the consequent.

The first and fourth terms of a proportion are called the *extremes*, and the second and third terms the *means*.

Thus in the proportion $a:b=c:d$, a and d are the extremes, and b and c the means.

307. In a proportion in which the means are equal, either mean is called a **Mean Proportional** between the first and last terms, and the last term is called a **Third Proportional** to the first and second terms.

A **Fourth Proportional** to three quantities is the fourth term of a proportion whose first three terms are the three quantities taken in their order.

Thus in the proportion $a:b=b:c$, b is a mean proportional between a and c, and c is a third proportional to a and b.

In the proportion $a:b=c:d$, d is a fourth proportional to a, b, and c.

308. A **Continued Proportion** is a series of equal ratios, in which each consequent is the same as the following antecedent; as,

$$a : b = b : c = c : d = d : e.$$

PROPERTIES OF PROPORTIONS.

309. *In any proportion the product of the extremes is equal to the product of the means.*

Let the proportion be $a : b = c : d$.

Then by Art. 305, $\frac{a}{b} = \frac{c}{d}$.

Clearing of fractions, $ad = bc$.

310. *A mean proportional between two quantities is equal to the square root of their product.*

Let the proportion be $a : b = b : c$.

Then by Art. 309, $b^2 = ac$.

Whence, $b = \sqrt{ac}$.

311. From the equation $ad = bc$, we obtain

$$a = \frac{bc}{d}, \text{ and } b = \frac{ad}{c}.$$

That is, in any proportion either extreme is equal to the product of the means divided by the other extreme; and either mean is equal to the product of the extremes divided by the other mean.

312. (Converse of Art. 309.) *If the product of two quantities is equal to the product of two others, one pair may be made the extremes, and the other the means, of a proportion.*

Let $ad = bc$.

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$, or $\frac{a}{b} = \frac{c}{d}$.

Whence, $a : b = c : d$.

In a similar manner we may prove that:

$$a : c = b : d,$$

$$b : d = a : c,$$

$$c : d = a : b, \text{ etc.}$$

313. *In any proportion the terms are in proportion by* **Alternation**; *that is, the first term is to the third, as the second term is to the fourth.*

Let $a : b = c : d.$

Then by Art. 309, $ad = bc.$

Whence by Art. 312, $a : c = b : d.$

314. *In any proportion the terms are in proportion by* **Inversion**; *that is, the second term is to the first, as the fourth term is to the third.*

Let $a : b = c : d.$

Then, $ad = bc.$

Whence, $b : a = d : c.$

315. *In any proportion the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the first term, as the sum of the last two terms is to the third term.*

Let $a : b = c : d.$

Then, $ad = bc.$

Adding both members to ac,

$$ac + ad = ac + bc,$$

or, $$a(c + d) = c(a + b).$$

Whence (Art. 312),

$$a + b : a = c + d : c.$$

Similarly we may prove that

$$a + b : b = c + d : d.$$

316. *In any proportion the terms are in proportion by* **Division**; *that is, the difference of the first two terms is to the first term, as the difference of the last two terms is to the third term.*

Let $$a : b = c : d.$$

Then, $$ad = bc.$$

Subtracting both members from ac,

$$ac - ad = ac - bc,$$

or, $$a(c - d) = c(a - b).$$

Whence, $$a - b : a = c - d : c.$$

Similarly, $$a - b : b = c - d : d.$$

317. *In any proportion the terms are in proportion by* **Composition and Division**; *that is, the sum of the first two terms is to their difference, as the sum of the last two terms is to their difference.*

Let $$a : b = c : d.$$

Then by Art. 315, $$\frac{a+b}{a} = \frac{c+d}{c}. \qquad (1)$$

And by Art. 316, $$\frac{a-b}{a} = \frac{c-d}{c}. \qquad (2)$$

Dividing (1) by (2), $$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Whence, $$a + b : a - b = c + d : c - d.$$

318. *In a series of equal ratios, any antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

Let $$a : b = c : d = e : f.$$

Then by Art. 309, $$ad = bc,$$

and $$af = be.$$

Also, $$ab = ba.$$

Adding, $$a(b + d + f) = b(a + c + e).$$

Whence (Art. 312), $$a : b = a + c + e : b + d + f.$$

319. *In any number of proportions, the products of the corresponding terms are in proportion.*

Let $a : b = c : d,$

and $e : f = g : h.$

Then, $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$.

Multiplying these equals,

$$\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}, \text{ or } \frac{ae}{bf} = \frac{cg}{dh}.$$

Whence, $ae : bf = cg : dh.$

320. *In any proportion, like powers or like roots of the terms are in proportion.*

Let $a : b = c : d.$

Then, $\frac{a}{b} = \frac{c}{d}.$

Therefore, $\frac{a^n}{b^n} = \frac{c^n}{d^n}.$

Whence, $a^n : b^n = c^n : d^n.$

In a similar manner we may prove that

$$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}.$$

321. *In any proportion, if the first two terms are multiplied by any quantity, as also the last two, the resulting quantities will be in proportion.*

Let $a : b = c : d.$

Then, $\frac{a}{b} = \frac{c}{d}.$

Therefore, $\frac{ma}{mb} = \frac{nc}{nd}.$

Whence, $ma : mb = nc : nd.$

In a similar manner we may prove that

$$\frac{a}{m}:\frac{b}{m}=\frac{c}{n}:\frac{d}{n}.$$

Note. Either m or n may be unity; that is, either couplet may be multiplied or divided without multiplying or dividing the other.

322. *In any proportion, if the first and third terms are multiplied by any quantity, as also the second and fourth terms, the resulting quantities will be in proportion.*

Let $$a:b=c:d.$$

Then, $$\frac{a}{b}=\frac{c}{d}.$$

Therefore, $$\frac{ma}{nb}=\frac{mc}{nd}.$$

Whence, $$ma:nb=mc:nd.$$

In a similar manner we may prove that

$$\frac{a}{m}:\frac{b}{n}=\frac{c}{m}:\frac{d}{n}.$$

Note. Either m or n may be unity.

323. *If three quantities are in continued proportion, the first is to the third as the square of the first is to the square of the second.*

Let $$a:b=b:c.$$

Then, $$\frac{a}{b}=\frac{b}{c}.$$

Therefore, $$\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}.$$

Or, $$\frac{a}{c}=\frac{a^2}{b^2}.$$

Whence, $$a:c=a^2:b^2.$$

324. *If four quantities are in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second.*

Let $$a : b = b : c = c : d.$$

Then, $$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$$

Therefore, $$\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}.$$

Or, $$\frac{a}{d} = \frac{a^3}{b^3}.$$

Whence, $$a : d = a^3 : b^3.$$

Note. The ratio $a^2 : b^2$ is called the *duplicate ratio,* and the ratio $a^3 : b^3$ the *triplicate ratio,* of $a : b$.

PROBLEMS.

325. 1. Solve the equation,

$$x + 1 : x - 1 = a + b : a - b.$$

By Art. 317, $$2x : 2 = 2a : 2b.$$

Whence by Art. 321, $$x : 1 = a : b.$$

Therefore, $$x = \frac{a}{b}, \quad \textit{Ans.}$$

2. If $x : y = (x + z)^2 : (y + z)^2$, prove that z is a mean proportional between x and y.

From the given proportion, by Art. 309,

$$y(x + z)^2 = x(y + z)^2.$$

Or, $$x^2y + 2xyz + yz^2 = xy^2 + 2xyz + xz^2.$$

Or, $$x^2y - xy^2 = xz^2 - yz^2.$$

Dividing by $x - y$, $$xy = z^2.$$

Therefore z is a mean proportional between x and y.

3. Find the first term of the proportion whose last three terms are 18, 6, and 27.

4. Find the second term of the proportion whose first, third, and fourth terms are 4, 20, and 55.

5. Find a fourth proportional to $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{2}{7}$.

6. Find a third proportional to $\frac{3}{4}$ and $\frac{5}{6}$.

7. Find a mean proportional between 8 and 18.

8. Find a mean proportional between 14 and 42.

9. Find a mean proportional between $2\frac{2}{9}$ and $\frac{5}{12}$.

Solve the following equations:

10. $2x-5:3x+2=x-1:7x+1.$

11. $x^2-4:x^2-9=x^2-5x+6:x^2+4x+3.$

12. $x+\sqrt{1-x^2}:x-\sqrt{1-x^2}=a+\sqrt{b^2-a^2}:a-\sqrt{b^2-a^2}.$

13. $\begin{cases} x:y=\ \ 3:5. \\ x:4=15:y. \end{cases}$ **14.** $\begin{cases} x+y:x-y=a+b:a-b. \\ x^2+y^2=a^2b^2(a^2+b^2). \end{cases}$

15. Find two numbers in the ratio of $2\frac{1}{2}$ to 2, such that when each is diminished by 5, they shall be in the ratio of $1\frac{1}{3}$ to 1.

16. Divide 50 into two parts such that the greater increased by 3 shall be to the less diminished by 3, as 3 to 2.

17. Divide 12 into two parts such that their product shall be to the sum of their squares as 3 to 10.

18. Find two numbers in the ratio of 4 to 9, such that 12 is a mean proportional between them.

19. The sum of two numbers is to their difference as 10 to 3, and their product is 364. What are the numbers?

20. If $a-b:b-c=b:c$, prove that b is a mean proportional between a and c.

21. If $5a + 4b : 9a + 2b = 5b + 4c : 9b + 2c$, prove that b is a mean proportional between a and c.

22. If $(a + b + c + d)(a - b - c + d) = (a - b + c - d)(a + b - c - d)$, prove that $a : b = c : d$.

23. If $ax - by : cx - dy = ay - bz : cy - dz$, prove that y is a mean proportional between x and z.

24. Find two numbers such that if 3 is added to each, they will be in the ratio of 4 to 3; and if 8 is subtracted from each, they will be in the ratio of 9 to 4.

25. There are two numbers whose product is 96, and the difference of their cubes is to the cube of their difference as 19 to 1. What are the numbers?

26. Divide $564 between A, B, and C, so that A's share may be to B's in the ratio of 5 to 9, and B's share to C's in the ratio of 7 to 10.

27. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; whereas, if it had been moving in the same direction with him, it would have passed him in 30 seconds. Compare the rates of the two trains.

28. Each of two vessels contains a mixture of wine and water. A mixture, consisting of equal measures from the two vessels, contains as much wine as water; and another mixture, consisting of four measures from the first vessel and one from the second, is composed of wine and water in the ratio of 2 to 3. Find the ratio of wine to water in each vessel.

29. Divide a into two parts such that the first increased by b shall be to the second diminished by b, as $a + 3b$ is to $a - 3b$.

XXIX. VARIATION.

326. One quantity is said to *vary directly* as another when the ratio of any two values of the first is equal to the ratio of the corresponding values of the second.

Note. It is customary to omit the word "directly," and say simply that one quantity *varies* as another.

327. Suppose, for example, that a workman receives a fixed sum per day.

The amount which he receives for m days will be to the amount which he receives for n days as m is to n; that is, the ratio of any two amounts received is equal to the ratio of the corresponding numbers of days worked.

Hence the amount which the workman receives *varies* as the number of days during which he works.

328. One quantity is said to *vary inversely* as another when the first varies directly as the *reciprocal* of the second.

Thus, the time in which a railway train will traverse a fixed route varies inversely as the speed; that is, if the speed is *doubled*, the train will traverse its route in *one-half* the time.

329. One quantity is said to vary as two others *jointly* when it varies directly as their product.

Thus, the wages of a workman varies jointly as the amount which he receives per day, and the number of days during which he works.

330. One quantity is said to vary directly as a second and inversely as a third, when it varies jointly as the second and the reciprocal of the third.

Thus, in physics, the attraction of a body varies directly as the quantity of matter, and inversely as the square of the distance.

331. The symbol $\propto$ is used to express variation; thus, $a \propto b$ is read "a varies as b."

332. *If $x \propto y$, then x is equal to y multiplied by a constant quantity.*

Let x' and y' denote a *fixed* pair of corresponding values of x and y, and x and y any other pair.

Then from the definition of Art. 326,

$$\frac{x}{x'} = \frac{y}{y'}, \text{ or } x = \frac{x'}{y'}y.$$

Denoting the constant ratio $\frac{x'}{y'}$ by m, we have

$$x = my.$$

333. It follows from Arts. 328, 329, 330, and 332 that:

1. *If x varies inversely as y, $x = \frac{m}{y}$.*
2. *If x varies jointly as y and z, $x = myz$.*
3. *If x varies directly as y and inversely as z, $x = \frac{my}{z}$.*

334. Problems in variation are readily solved by converting the variation into an equation by aid of Arts. 332 or 333.

EXAMPLES.

335. 1. If x varies inversely as y, and is equal to 9 when $y = 8$, what is the value of x when $y = 18$?

If x varies inversely as y, we have by Art. 333,

$$x = \frac{m}{y}.$$

Putting $x = 9$ and $y = 8$, we obtain

$$9 = \frac{m}{8}, \text{ or } m = 72.$$

Whence, $$x = \frac{72}{y}.$$

Hence, if $y = 18$, we have $x = \frac{72}{18} = 4$, *Ans.*

2. Given that the area of a triangle varies jointly as its base and altitude, what will be the base of a triangle whose altitude is 12, equivalent to the sum of two triangles whose bases are 10 and 6, and altitudes 3 and 9, respectively?

Let B, H, and A denote the base, altitude, and area, respectively, of any triangle, and B' the base of the required triangle.

Then since A varies jointly as B and H, we have

$$A = mBH \text{ (Art. 333).}$$

Therefore the area of the first triangle is $m \times 10 \times 3$, or $30m$, and the area of the second is $m \times 6 \times 9$, or $54m$.

Hence the area of the required triangle is

$$30m + 54m, \text{ or } 84m.$$

But the area of the required triangle is also $m \times B' \times 12$.

Therefore, $\quad 12mB' = 84m.$

Whence, $\quad B' = 7$, *Ans.*

3. If $y \propto x$, and is equal to 36 when $x = 4$, what is its value when $x = 7$?

4. If $y \propto z^2$, and is equal to 15 when $z = 3$, what is the value of y in terms of z^2?

5. If x varies inversely as y, and is equal to 4 when $y = 2$, what is the value of y when $x = \frac{4}{3}$?

6. If z varies jointly as x and y, and is equal to 90 when $x = 3$ and $y = 6$, what is the value of z when $x = 2$ and $y = 7$?

7. If x varies directly as y and inversely as z, and is equal to 4 when $y = 2$ and $z = 3$, what is the value of x when $y = 35$ and $z = 15$?

8. If $2x - 3 \propto 3y + 7$, and $x = 3$ when $y = 1$, what is the value of x when $y = -1$?

9. If $x^3 \propto y^2$, and $x = 6$ when $y = 3$, what is the value of y when $x = 2$?

10. Two quantities vary directly and inversely as x, respectively. If their sum is equal to 7 when $x = 2$, and to -13 when $x = -3$, what are the quantities?

11. Given that the volume of a pyramid varies jointly as its base and altitude, what will be the altitude of a pyramid whose base is 12, equivalent to the sum of two pyramids whose bases are 5 and 8, and altitudes 12 and 6, respectively?

12. If the illumination from a source of light varies inversely as the square of the distance, how much farther from a candle must a book, which is now 3 inches off, be removed so as to receive just half as much light?

13. Two circular plates of gold, each an inch thick, the diameters of which are 6 and 8 inches, respectively, are melted and formed into a single circular plate one inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

14. Three spheres of lead whose diameters are 3, 4, and 5 inches, respectively, are melted and formed into a single sphere. Find its diameter, having given that the volume of a sphere varies as the cube of its diameter.

15. If 5 men in 6 weeks earn $57, how many weeks will it take 4 men to earn $76; it being given that the amount earned varies jointly as the number of men, and the number of weeks during which they work.

16. If the volume of a cylinder of revolution varies jointly as its altitude and the square of its radius, what will be the radius of a cylinder, whose altitude is 18, equivalent to the sum of two cylinders whose altitudes are 5 and 12, and radii 6 and 9, respectively?

17. Given that y is equal to the sum of two quantities, of which one is constant and the other varies as xy. If y is equal to 1 when $x = -2$, and to $-\frac{7}{8}$ when $x = 2$, what is the expression for y in terms of x?

XXX. ARITHMETICAL PROGRESSION.

336. An **Arithmetical Progression** is a series of terms, each of which is derived from the preceding by adding a constant quantity called the *common difference*.

Thus, 1, 3, 5, 7, 9, 11, ... is an increasing arithmetical progression, in which the common difference is 2.

Again, 12, 9, 6, 3, 0, −3, ... is a decreasing arithmetical progression, in which the common difference is −3.

337. *Given the first term, a, the common difference, d, and the number of terms, n, to find the last term, l.*

The progression is

$$a,\ a+d,\ a+2d,\ a+3d, \cdots$$

It will be observed that the coefficient of d in any term is one less than the number of the term. Hence, in the nth, or last term, the coefficient of d will be $n-1$. That is,

$$l = a + (n-1)d. \qquad \text{(I.)}$$

338. *Given the first term, a, the last term, l, and the number of terms, n, to find the sum of the series, S.*

$$S = a + (a+d) + (a+2d) + \cdots + (l-d) + l.$$

Writing the series in reverse order,

$$S = l + (l-d) + (l-2d) + \cdots + (a+d) + a.$$

Adding these equations, term by term,

$$2S = (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l)$$
$$= n(a+l).$$

Therefore, $$S = \frac{n}{2}(a+l). \qquad \text{(II.)}$$

339. Substituting in (II.) the value of l from (I.), we have

$$S = \frac{n}{2}[2a + (n-1)d].$$

EXAMPLES.

340. 1. In the series 8, 5, 2, -1, -4, $\cdots$ to 27 terms, find the last term and the sum.

In this case, $a = 8$, $d = -3$, $n = -27$.

Substituting in (I.) and (II.),

$$l = 8 + (27 - 1)(-3) = 8 - 78 = - \quad 70.$$

$$S = \frac{27}{2}(8 - 70) = 27 \times (-31) = -837.$$

Note. The common difference may be found by subtracting the first term from the second. Thus, in the series

$$\frac{5}{3}, \ -\frac{1}{6}, \ -2, \ \cdots, \text{ we have } d = -\frac{1}{6} - \frac{5}{3} = -\frac{11}{6}.$$

In each of the following, find the last term and the sum of the series:

2. 1, 6, 11, $\cdots$ to 15 terms.

3. 7, 3, -1, $\cdots$ to 20 terms.

4. -9, -6, -3, $\cdots$ to 23 terms.

5. -5, -10, -15, $\cdots$ to 29 terms.

6. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \cdots$ to 35 terms.

7. $\frac{3}{5}, \frac{8}{15}, \cdots$ to 19 terms.

8. $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \cdots$ to 16 terms.

9. $\frac{1}{2}, \frac{5}{11}, \cdots$ to 22 terms.

10. $-3, -\frac{5}{2}, \cdots$ to 17 terms.

11. $-\frac{2}{5}, \frac{1}{3}, \cdots$ to 14 terms.

341. If any three of the five elements of an arithmetical progression are given, the other two may be found by substituting the given values in the fundamental formulæ (I.) and (II.), and solving the resulting equations.

1. Given $a = -\frac{5}{3}$, $n = 20$, $S = -\frac{5}{3}$; find d and l.

Substituting the given values in (I.) and (II.), we have

$$l = -\frac{5}{3} + 19d. \qquad (1)$$

$$-\frac{5}{3} = 10\left(-\frac{5}{3} + l\right); \text{ or, } -\frac{1}{6} = -\frac{5}{3} + l. \qquad (2)$$

From (2), $l = \frac{5}{3} - \frac{1}{6} = \frac{3}{2}$.

Substituting in (1),

$$\frac{3}{2} = -\frac{5}{3} + 19d; \text{ or, } d = \frac{1}{6}.$$

Ans. $d = \frac{1}{6}$, $l = \frac{3}{2}$.

2. Given $d = -3$, $l = -39$, $S = -264$; find a and n.

Substituting in (I.) and (II.),

$$-39 = a + (n-1)(-3), \text{ or } a = 3n - 42. \qquad (1)$$

$$-264 = \frac{n}{2}(a - 39), \text{ or } an - 39n = -528. \qquad (2)$$

Substituting the value of a from (1) in (2),

$$3n^2 - 42n - 39n = -528,$$

or,

$$n^2 - 27n = -176.$$

Whence, $$n = \frac{27 \pm \sqrt{729 - 704}}{2} = \frac{27 \pm 5}{2} = 16 \text{ or } 11.$$

Substituting in (1),

$$a = 48 - 42, \text{ or } 33 - 42 = 6 \text{ or } -9.$$

Ans. $a = 6$, $n = 16$; or, $a = -9$, $n = 11$.

Note. The interpretation of the two answers is as follows:

If $a = 6$, and $n = 16$, the series is

$6,\ 3,\ 0,\ -3,\ -6,\ -9,\ -12,\ -15,\ -18,\ -21,\ -24, -27, -30,$
$-33,\ -36,\ -39.$

If $a = -9$, and $n = 11$, the series is

$-9, -12, -15, -18, -21, -24, -27, -30, -33, -36, -39.$

In each of these the last term is -39, and the sum is -264.

3. Given $a = \frac{1}{3}$, $d = -\frac{1}{12}$, $S = -\frac{3}{2}$; find l and n.

Substituting in (I.) and (II.),

$$l = \frac{1}{3} + (n-1)\left(-\frac{1}{12}\right),\ \text{or}\ l = \frac{5-n}{12}. \qquad (1)$$

$$-\frac{3}{2} = \frac{n}{2}\left(\frac{1}{3} + l\right), \quad \text{or} \quad n + 3ln = -9. \qquad (2)$$

Substituting the value of l from (1) in (2),

$$n + \frac{5n - n^2}{4} = -9, \ \text{or} \ n^2 - 9n = 36.$$

Solving this equation, $n = 12$ or -3.

The second value is inapplicable, for the number of terms in a progression must be a *positive integer*.

Substituting the value $n = 12$ in (1),

$$l = \frac{5-12}{12} = -\frac{7}{12}.$$

Ans. $l = -\frac{7}{12}$, $n = 12$.

Note. A negative or fractional value of n is inapplicable, and should be rejected together with all other values dependent upon it.

EXAMPLES.

4. Given $d = 4$, $l = 75$, $n = 19$; find a and S.

5. Given $d = -1$, $n = 15$, $S = -\frac{165}{2}$; find a and l.

6. Given $a = -\frac{2}{3}$, $n = 18$, $l = 5$; find d and S.

7. Given $a = -\frac{3}{4}$, $n = 7$, $S = -7$; find d and l.

8. Given $a = \frac{3}{2}$, $l = -\frac{57}{2}$, $S = -\frac{351}{2}$; find d and n.

9. Given $l = -31$, $n = 13$, $S = -169$; find a and d.

10. Given $d = -3$, $S = -328$, $a = 2$; find l and n.

11. Given $a = 3$, $l = 42\frac{2}{3}$, $d = 2\frac{1}{3}$; find n and S.

12. Given $d = -4$, $n = 17$, $S = -493$; find a and l.

13. Given $l = \frac{7}{2}$, $d = \frac{1}{3}$, $S = 20$; find a and n.

14. Given $l = \frac{79}{2}$, $n = 21$, $S = \frac{819}{2}$; find a and d.

15. Given $a = -\frac{1}{3}$, $l = -\frac{4}{3}$, $S = -\frac{40}{3}$; find d and n.

16. Given $a = -\frac{3}{4}$, $n = 15$, $S = 120$; find d and l.

17. Given $l = -47$, $d = -1$, $S = -1118$; find a and n.

18. Given $a = 6$, $d = -\frac{5}{3}$, $S = -\frac{203}{3}$; find n and l.

From (I.) and (II.) general formulæ for the solution of cases like the above may be readily derived.

19. Given a, d, and S; derive the formula for n.

Substituting the value of l from (I.) in (II.),

$$2S = n[2a + (n-1)d], \text{ or } dn^2 + (2a-d)n = 2S.$$

This is a quadratic in n, and may be solved by the method of Art. 265.

Multiplying by $4d$, and adding $(2a-d)^2$ to both members,

$$4d^2n^2+4d(2a-d)n+(2a-d)^2=8dS+(2a-d)^2.$$

Extracting the square root,

$$2dn+2a-d=\pm\sqrt{8dS+(2a-d)^2}.$$

Whence,
$$n=\frac{d-2a\pm\sqrt{8dS+(2a-d)^2}}{2d}.$$

20. Given a, l, and n; derive the formula for d.

21. Given a, n, and S; derive the formulæ for d and l.

22. Given d, n, and S; derive the formulæ for a and l.

23. Given a, d, and l; derive the formulæ for n and S.

24. Given d, l, and n; derive the formulæ for a and S.

25. Given l, n, and S; derive the formulæ for a and d.

26. Given a, d, and S; derive the formula for l.

27. Given a, l, and S; derive the formulæ for d and n.

28. Given d, l, and S; derive the formulæ for a and n.

342. *To insert any number of arithmetical means between two given terms.*

For example, let it be required to insert 5 arithmetical means between 3 and -5.

This signifies that we are to find an arithmetical progression of 7 terms, whose first term is 3, and last term -5.

Substituting $a=3$, $l=-5$, and $n=7$ in (I.), we have

$$-5=3+6d,\text{ or }d=-\frac{4}{3}.$$

Hence the required series is

$$3,\ \frac{5}{3},\ \frac{1}{3},\ -1,\ -\frac{7}{3},\ -\frac{11}{3},\ -5.$$

343. Let x denote the arithmetical mean between a and b.

Then, by the nature of the progression,

$$x - a = b - x, \text{ or } 2x = a + b.$$

Whence, $$x = \frac{a+b}{2}.$$

That is, *the arithmetical mean between two quantities is equal to one-half their sum.*

EXAMPLES.

344. **1**. Insert 5 arithmetical means between 2 and 4.

2. Insert 7 arithmetical means between 3 and -1.

3. Insert 4 arithmetical means between -1 and -7.

4. Insert 6 arithmetical means between -8 and -4.

5. Insert 8 arithmetical means between $\frac{1}{2}$ and $-\frac{13}{10}$.

Find the arithmetical mean between:

6. $2\frac{1}{3}$ and $-1\frac{4}{5}$.

7. $(a+b)^2$ and $-(a-b)^2$.

8. $\dfrac{a+b}{a-b}$ and $\dfrac{a-b}{a+b}$.

PROBLEMS.

345. **1**. The sixth term of an arithmetical progression is $\frac{5}{6}$, and the fifteenth term is $\frac{16}{3}$. Find the first term.

By Art. 337, the sixth term is $a + 5d$, and the fifteenth term is $a + 14d$; hence,

$$\begin{cases} a + 5d = \dfrac{5}{6}. & (1) \\ a + 14d = \dfrac{16}{3}. & (2) \end{cases}$$

Subtracting (1) from (2), $9d = \frac{9}{2}$, or $d = \frac{1}{2}$.

Substituting in (2), $a + 7 = \frac{16}{3}$; whence, $a = -\frac{5}{3}$, *Ans.*

2. Find four quantities in arithmetical progression such that the product of the extremes shall be 45, and the product of the means 77.

Let the quantities be $x - 3y$, $x - y$, $x + y$, and $x + 3y$. Then, by the conditions,

$$\begin{cases} x^2 - 9y^2 = 45. \\ x^2 - y^2 = 77. \end{cases}$$

Solving these equations, $x = \pm 9$ and $y = \pm 2$.

Therefore the quantities are 3, 7, 11, and 15; or, -3, -7, -11, and -15.

Note. In problems like the above it is convenient to represent the unknown quantities by *symmetrical* expressions. Thus if five quantities had been required, we should have represented them by $x - 2y$, $x - y$, x, $x + y$, and $x + 2y$.

3. Find the sum of the odd numbers from 1 to 100.

4. The seventh term of an arithmetical progression is 27, and the thirteenth term is -3. Find the twenty-first term.

5. Find four numbers in arithmetical progression such that the sum of the first and third shall be 22, and the sum of the second and fourth 36.

6. A person saves $270 the first year, $245 the second, and so on. In how many years will a person who saves every year $145 have saved as much as he?

7. In the progression m, $2m - 3n$, $3m - 6n$, ... to 10 terms, find the last term and the sum of the series.

8. The seventh term of an arithmetical progression is $5a + 4b$, and the nineteenth term is $9a - 2b$. Find the fifteenth term.

9. Find the sum of the even numbers beginning with 2 and ending with 500.

10. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

11. The seventh term of an arithmetical progression is $-\frac{1}{2}$, the thirteenth term is $\frac{3}{2}$, and the last term is $\frac{9}{2}$. Find the number of terms.

12. Find five quantities in arithmetical progression such that the sum of the first, third, and fourth is 3, and the product of the second and fifth is -8.

13. Two persons start together. One travels 10 leagues a day; the other 8 leagues the first day, which he augments daily by half a league. After how many days, and at what distance from the point of departure, will they come together?

14. A body falls $16\frac{1}{12}$ feet the first second, and in each succeeding second $32\frac{1}{6}$ feet more than in the next preceding one. How far will it fall in 16 seconds?

15. Find three quantities in arithmetical progression such that the sum of the squares of the first and third exceeds the second by 123, and the second exceeds one-third the first by 6.

16. After A had travelled $2\frac{3}{4}$ hours at the rate of 4 miles an hour, B set out to overtake him, and went $4\frac{1}{2}$ miles the first hour, $4\frac{3}{4}$ the second, 5 the third, and so on, increasing his speed a quarter of a mile every hour. In how many hours would he overtake A?

17. If a person should save \$100 a year, and put this sum at simple interest at 5 per cent at the end of each year, to how much would his property amount at the end of 20 years?

18. The digits of a number of three figures are in arithmetical progression; the first digit exceeds the sum of the second and third by 1; and if 594 be subtracted from the number, the digits will be inverted. Find the number.

XXXI. GEOMETRICAL PROGRESSION.

346. A **Geometrical Progression** is a series of terms, each of which is derived from the preceding by multiplying by a constant quantity called the *ratio*.

Thus, 2, 6, 18, 54, 162, ... is an increasing geometrical progression in which the ratio is 3.

Again, 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, ... is a decreasing geometrical progression in which the ratio is $\frac{1}{3}$.

Negative values of the ratio are also admissible; thus, -3, 6, -12, 24, -48, ... is a geometrical progression in which the ratio is -2.

347. *Given the first term, a, the ratio, r, and the number of terms, n, to find the last term, l.*

The progression is a, ar, ar^2, ar^3, ...

It will be observed that the exponent of r in any term is one less than the number of the term. Hence, in the nth or last term, the exponent of r will be $n-1$. That is,

$$l = ar^{n-1}. \qquad \text{(I.)}$$

348. *Given the first term, a, the last term, l, and the ratio, r, to find the sum of the series, S.*

$$S = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}.$$

Multiplying each term by r,

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n.$$

Subtracting the first equation from the second,

$$rS - S = ar^n - a; \text{ or, } S = \frac{ar^n - a}{r-1}.$$

But from (I.), Art. 347, $rl = ar^n$. Hence,

$$S = \frac{rl - a}{r-1}. \qquad \text{(II.)}$$

EXAMPLES.

349. 1. In the series $3, 1, \frac{1}{3}, \cdots$ to 7 terms, find the last term and the sum.

In this case, $a = 3$, $r = \frac{1}{3}$, $n = 7$. Substituting in (I.) and (II.),

$$l = 3\left(\frac{1}{3}\right)^6 = \frac{1}{3^5} = \frac{1}{243}.$$

$$S = \frac{\frac{1}{3} \times \frac{1}{243} - 3}{\frac{1}{3} - 1} = \frac{\frac{1}{729} - 3}{-\frac{2}{3}} = \frac{-\frac{2186}{729}}{-\frac{2}{3}} = \frac{1093}{243}.$$

Note. The ratio may be found by dividing the second term by the first.

2. In the series $-2, 6, -18, 54, \ldots$ to 8 terms, find the last term and the sum.

In this case, $a = -2$, $r = \frac{6}{-2} = -3$, $n = 8$. Hence,

$$l = -2(-3)^7 = -2 \times (-2187) = 4374.$$

$$S = \frac{-3 \times 4374 - (-2)}{-3 - 1} = \frac{-13122 + 2}{-4} = 3280.$$

In each of the following, find the last term and the sum of the series:

3. $1, 2, 4, \cdots$ to 9 terms.

4. $3, 2, \frac{4}{3}, \cdots$ to 7 terms.

5. $-2, 8, -32, \cdots$ to 6 terms.

6. $2, -1, \frac{1}{2}, \cdots$ to 10 terms.

7. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ to 11 terms.

8. $\frac{2}{3}, -1, \frac{3}{2}, \cdots$ to 8 terms.

9. 8, 4, 2, $\cdots$ to 9 terms.

10. $\frac{3}{4}, -\frac{1}{4}, \frac{1}{12}, \cdots$ to 6 terms.

11. 3, -6, 12, $\cdots$ to 7 terms.

12. $-\frac{2}{3}, -\frac{1}{3}, -\frac{1}{6}, \cdots$ to 10 terms.

350. If any three of the five elements of a geometrical progression are given, the other two may be found by substituting the given values in the fundamental formulæ (I.) and (II.), and solving the resulting equations.

But in certain cases the operation involves the solution of an equation of a degree higher than the second; and in others the unknown quantity appears as an exponent, the solution of which form of equation can usually only be effected by aid of logarithms (Art. 427).

In all such examples in the present chapter, the equations may be solved by inspection.

1. Given $a = -2$, $n = 5$, $l = -32$; find r and S.

Substituting the given values in (I.), we have

$$-32 = -2r^4; \text{ whence, } r^4 = 16, \text{ or } r = \pm 2.$$

Substituting in (II.),

$$\text{If } r = 2,\ S = \frac{2(-32) - (-2)}{2-1} = -64 + 2 = -62.$$

$$\text{If } r = -2,\ S = \frac{(-2)(-32) - (-2)}{-2-1} = \frac{64+2}{-3} = -22.$$

Ans. $r = 2$, $S = -62$; or, $r = -2$, $S = -22$.

Note. The interpretation of the two answers is as follows:

If $r = 2$, the series is $-2, -4, -8, -16, -32$, in which the sum is -62.
If $r = -2$, the series is $-2, 4, -8, 16, -32$, in which the sum is -22.

2. Given $a=3$, $r=-\frac{1}{3}$, $S=\frac{1640}{729}$; find n and l.

Substituting in (II.),

$$\frac{1640}{729}=\frac{-\frac{1}{3}l-3}{-\frac{1}{3}-1}=\frac{l+9}{4}.$$

Whence, $\quad l+9=\frac{6560}{729}$; or, $l=-\frac{1}{729}$.

Substituting in (I.),

$$-\frac{1}{729}=3\left(-\frac{1}{3}\right)^{n-1}; \text{ or, } \left(-\frac{1}{3}\right)^{n-1}=-\frac{1}{2187}.$$

Whence, by inspection,

$$n-1=7, \text{ or } n=8.$$

EXAMPLES.

3. Given $r=2$, $n=10$, $l=256$; find a and S.

4. Given $r=-2$, $n=6$, $S=\frac{63}{2}$; find a and l.

5. Given $a=2$, $n=7$, $l=1458$; find r and S.

6. Given $a=1$, $r=3$, $l=81$; find n and S.

7. Given $r=\frac{1}{3}$, $n=8$, $S=\frac{6560}{6561}$; find a and l.

8. Given $a=3$, $n=6$, $l=-\frac{3}{1024}$; find r and S.

9. Given $a=2$, $l=\frac{1}{32}$, $S=\frac{127}{32}$; find n and r.

10. Given $a=\frac{1}{2}$, $r=-3$, $S=-91$; find n and l.

11. Given $l=-128$, $r=2$, $S=-255$; find a and n.

From (I.) and (II.) general formulæ may be derived for the solution of cases like the above.

12. Given a, r, and S; derive the formula for l.

13. Given a, l, and S; derive the formula for r.

14. Given r, l, and S; derive the formula for a.

15. Given r, n, and l; derive the formulæ for a and S.

16. Given r, n, and S; derive the formulæ for a and l.

17. Given a, n, and l; derive the formulæ for r and S.

Note. If the given elements are n, l, and S, *equations* for a and r may be found, but there are no definite *formulæ* for their values. The same is the case when the given elements are a, n, and S.

The general formulæ for n involve logarithms; these cases are discussed in Art. 427.

351. The limit (Art. 297) to which the sum of the terms of a *decreasing* geometrical progression approaches, as the number of terms increases indefinitely, is called the *sum of the series to infinity.*

The value of S in formula (II.), Art. 348, may be written

$$S=\frac{a-rl}{1-r}.$$

In a decreasing geometrical progression, the greater the number of terms taken, the smaller will be the value of the last term.

Hence as the number of terms increases indefinitely, the term rl approaches the limit 0.

Therefore the fraction $\frac{a-rl}{1-r}$ approaches the limit $\frac{a}{1-r}$.

That is, the sum of a decreasing geometrical progression to infinity is given by the formula

$$S=\frac{a}{1-r}. \qquad \text{(III.)}$$

EXAMPLES.

1. Find the sum of the series $4, -\frac{8}{3}, \frac{16}{9}, \dots$ to infinity.

In this case, $a = 4,\ r = -\frac{2}{3}.$

Substituting in (III.), $S = \dfrac{4}{1 + \frac{2}{3}} = \dfrac{12}{5}$, *Ans.*

Find the sum of the following to infinity:

2. $2, 1, \frac{1}{2}, \dots$
3. $4, -2, 1, \dots$
4. $-1, \frac{1}{3}, -\frac{1}{9}, \dots$
5. $-3, -\frac{3}{5}, -\frac{3}{25}, \dots$
6. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \dots$
7. $3, -\frac{3}{10}, \frac{3}{100}, \dots$
8. $-8, -\frac{2}{5}, -\frac{1}{50}, \dots$
9. $1, -\frac{a^2}{x^2}, \frac{a^4}{x^4}, \dots$

352. *To find the value of a repeating decimal.*

This is a case of finding the sum of a geometrical progression to infinity, and may be solved by the formula of Art. 351.

1. Find the value of $.85151 \cdots$

$$.85151 \cdots = .8 + .051 + .00051 + \cdots$$

The terms after the first constitute a decreasing geometrical progression in which $a = .051$, and $r = .01$.

Substituting in (III.),

$$S = \frac{.051}{1 - .01} = \frac{.051}{.99} = \frac{51}{990} = \frac{17}{330}.$$

Hence the value of the given decimal is

$$\frac{8}{10} + \frac{17}{330} = \frac{281}{330},\ \textit{Ans.}$$

EXAMPLES.

Find the values of the following:

2. .7272...	**4.** .7333...	**6.** .110303...
3. .407407...	**5.** .52121...	**7.** .215454...

353. *To insert any number of geometrical means between two given terms.*

Example. Insert 4 geometrical means between 2 and $\frac{64}{243}$.

This signifies that we are to find a geometrical progression of 6 terms, whose first term is 2, and last term $\frac{64}{243}$.

Substituting $a = 2$, $n = 6$, and $l = \frac{64}{243}$ in (I.), we have

$$\frac{64}{243} = 2r^5\text{; whence, } r^5 = \frac{32}{243}, \text{ and } r = \frac{2}{3}.$$

Hence the required series is

$$2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \frac{32}{81}, \frac{64}{243}.$$

354. Let x denote the geometrical mean between a and b.

Then, by the nature of the progression,

$$\frac{x}{a} = \frac{b}{x}, \text{ or } x^2 = ab.$$

Whence, $$x = \sqrt{ab}.$$

That is, *the geometrical mean between two quantities is equal to the square root of their product.*

EXAMPLES.

355. **1.** Insert 6 geometrical means between 3 and $\frac{128}{729}$.

2. Insert 5 geometrical means between $\frac{1}{2}$ and $364\frac{1}{2}$.

3. Insert 6 geometrical means between -2 and -4374.

4. Insert 7 geometrical means between $\frac{3}{2}$ and $\frac{3}{512}$.

5. Insert 5 geometrical means between -2 and -128.

6. Insert 4 geometrical means between 3 and $-\frac{729}{1024}$.

Find the geometrical mean between:

7. $11\frac{2}{3}$ and $2\frac{1}{7}$.

8. $4x^2 + 12xy + 9y^2$ and $4x^2 - 12xy + 9y^2$.

9. $\dfrac{a^2 - ab}{ab + b^2}$ and $\dfrac{a^2 + ab}{ab - b^2}$.

PROBLEMS.

356. 1. Find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84.

Let the quantities be a, ar, and ar^2; then, by the conditions,

$$\begin{cases} a + ar + ar^2 = 14. & (1) \\ a^2 + a^2r^2 + a^2r^4 = 84. & (2) \end{cases}$$

Dividing (2) by (1), $\quad a - ar + ar^2 = 6. \quad (3)$

Subtracting (3) from (1), $\quad 2ar = 8$, or $r = \frac{4}{a}$. $\quad (4)$

Substituting in (1), $\quad a + 4 + \frac{16}{a} = 14.$

Or, $\quad a^2 - 10a = -16.$

Solving this equation, $\quad a = 8$ or 2.

Substituting in (4), $\quad r = \frac{4}{8}$ or $\frac{4}{2} = \frac{1}{2}$ or 2.

Therefore, the numbers are 2, 4, and 8.

2. The fifth term of a geometrical progression is 48, and the eighth term is -384. Find the first term.

3. The sum of the first and second of four quantities in geometrical progression is 15, and the sum of the third and fourth is 60. What are the quantities?

4. Find three quantities in geometrical progression, such that the sum of the first and second is 20, and the third exceeds the second by 30.

5. The fourth term of a geometrical progression is -108, and the eighth term is -8748. Find the first term.

6. A person who saved every year half as much again as he saved the previous year, had in seven years saved $2059. How much did he save the first year?

7. The elastic power of a ball, which falls from a height of a hundred feet, causes it to rise to 0.9375 of the height from which it fell, and to continue in this way diminishing the height to which it will rise, in geometrical progression, until it comes to rest. How far will it have moved?

8. The sum of four quantities in geometrical progression is 30, and the quotient of the fourth quantity divided by the sum of the second and third is $\frac{4}{3}$. Find the quantities.

9. The third term of a geometrical progression is $\frac{1}{24}$, and the sixth term is $\frac{9}{512}$. Find the eighth term.

10. Divide the number 39 into three parts in geometrical progression, such that the third part shall exceed the first by 24.

11. The product of three numbers in geometrical progression is 64, and the sum of the squares of the first and third is 68. What are the numbers?

12. The product of three quantities in geometrical progression is 8, and the sum of their cubes is 73. What are the quantities?

XXXII. HARMONICAL PROGRESSION.

357. Quantities are said to be in **Harmonical Progression** when their reciprocals form an arithmetical progression.

Thus, $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$ are in harmonical progression, because their reciprocals $1, 3, 5, 7, 9, \ldots$ form an arithmetical progression.

358. Any problem in harmonical progression, which is susceptible of solution, may be solved by taking the reciprocals of the terms and applying the formulæ of the arithmetical progression.

There will be found, however, no general formula for the *sum of the terms* of a harmonical series.

359. Let x denote the harmonical mean between a and b.

Then, by the nature of the progression, $\frac{1}{x}$ is the arithmetical mean between $\frac{1}{a}$ and $\frac{1}{b}$.

Whence, $$\frac{1}{x}=\frac{\frac{1}{a}+\frac{1}{b}}{2}\text{ (Art. 343) }=\frac{a+b}{2ab}.$$

Therefore, $$x=\frac{2ab}{a+b}.$$

360. *If any three consecutive terms of a harmonical series are taken, the first is to the third as the first minus the second is to the second minus the third.*

Let the terms be a, b, and c.

Then since $\frac{1}{a}, \frac{1}{b}$, and $\frac{1}{c}$ are in arithmetical progression,

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a},$$

or, $$\frac{b-c}{bc}=\frac{a-b}{ab}.$$

Multiplying both members by $\frac{ab}{b-c}$, we have

$$\frac{a}{c}=\frac{a-b}{b-c}.$$

EXAMPLES.

361. 1. In the series $2, \frac{2}{3}, \frac{2}{5}, \ldots$ to 36 terms, find the last term.

Taking the reciprocals of the terms, we have the arithmetical progression

$$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

In this case $a=\frac{1}{2}$, $d=1$, $n=36$.

Substituting in (I.), Art. 337, we have

$$l=\frac{1}{2}+(36-1)\times 1=\frac{71}{2}.$$

Taking the reciprocal of this, we obtain $\frac{2}{71}$ as the last term of the given harmonical series.

2. Insert 5 harmonical means between 2 and -3.

Taking the reciprocals of the terms, we have to insert 5 arithmetical means between $\frac{1}{2}$ and $-\frac{1}{3}$.

Substituting $a=\frac{1}{2}$, $l=-\frac{1}{3}$, and $n=7$, in (I.), Art. 337, we have

$$-\frac{1}{3}=\frac{1}{2}+6d; \text{ or, } d=-\frac{5}{36}.$$

Then the arithmetical series is

$$\frac{1}{2}, \frac{13}{36}, \frac{2}{9}, \frac{1}{12}, -\frac{1}{18}, -\frac{7}{36}, -\frac{1}{3}.$$

Therefore the required harmonical series is

$$2, \frac{36}{13}, \frac{9}{2}, 12, -18, -\frac{36}{7}, -3.$$

Find the last terms of the following:

3. $\frac{3}{4}, \frac{3}{11}, \frac{1}{6}, \ldots$ to 11 terms.

4. $\frac{1}{2}, -\frac{1}{3}, -\frac{1}{8}, \ldots$ to 17 terms.

5. $\frac{3}{2}, \frac{5}{3}, \frac{15}{8}, \ldots$ to 23 terms.

6. $-\frac{4}{3}, -\frac{3}{2}, -\frac{12}{7}, \ldots$ to 26 terms.

7. $-\frac{3}{7}, -\frac{6}{23}, -\frac{3}{16}, \ldots$ to 31 terms.

8. Insert 7 harmonical means between $\frac{2}{5}$ and $\frac{3}{10}$.

9. Insert 4 harmonical means between -2 and -8.

10. Insert 6 harmonical means between 3 and -1.

Find the harmonical mean between:

11. 3 and -5.

12. $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$.

13. Find the last term of the harmonical series

$a, b, \ldots$ to n terms.

14. If m harmonical means are inserted between a and b, what is the second mean?

15. The fourth and ninth terms of a harmonical progression are $-\frac{3}{7}$ and $-\frac{1}{9}$, respectively. What is the seventh term?

16. Prove that the geometrical mean between two quantities is a mean proportional between their arithmetical and harmonical means.

XXXIII. THE BINOMIAL THEOREM.

POSITIVE INTEGRAL EXPONENT.

362. The **Binomial Theorem** is a formula by means of which any power of a binomial may be expanded into a series.

Examples of its application have been given in Art. 196.

Proof of the Theorem for a Positive Integral Exponent.

363. If we *assume* the laws of Art. 196 to hold for the expansion of $(a+x)^n$, where n is any positive integer:

The exponent of a in the first term is n, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, n; multiplying n, the coefficient of the second term, by $n-1$, the exponent of a in that term, and dividing the result by the exponent of x increased by 1, or 2, we have $\frac{n(n-1)}{1\cdot 2}$ as the coefficient of the third term; and so on.

$$\text{Thus, } (a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots \quad (1)$$

This result is called the *Binomial Theorem*.

Multiplying both members by $a+x$, we have

$$\begin{aligned}(a+x)^{n+1} = a^{n+1} &+ na^n x + \frac{n(n-1)}{1\cdot 2}a^{n-1}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots \\ &+ a^n x + na^{n-1}x^2 + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^3 + \cdots\end{aligned}$$

Collecting the terms which contain like powers of a and x,

$$(a+x)^{n+1} = a^{n+1} + (n+1)a^n x + \left[\frac{n(n-1)}{1\cdot 2} + n\right]a^{n-1}x^2$$

$$+\left[\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \frac{n(n-1)}{1\cdot 2}\right]a^{n-2}x^3 + \cdots$$

$$= a^{n+1} + (n+1)a^n x + n\left[\frac{n-1}{2} + 1\right]a^{n-1}x^2$$

$$+\frac{n(n-1)}{1\cdot 2}\left[\frac{n-2}{3} + 1\right]a^{n-2}x^3 + \cdots$$

$$= a^{n+1} + (n+1)a^n x + n\left[\frac{n+1}{2}\right]a^{n-1}x^2$$

$$+\frac{n(n-1)}{1\cdot 2}\left[\frac{n+1}{3}\right]a^{n-2}x^3 + \cdots$$

$$= a^{n+1} + (n+1)a^n x + \frac{(n+1)n}{1\cdot 2}a^{n-1}x^2$$

$$+\frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}a^{n-2}x^3 + \cdots$$

It will be observed that this result is in accordance with the laws of Art. 196.

Hence, if the laws of Art. 196 hold for any power of $a+x$ whose exponent is a positive integer, they also hold for an exponent greater by 1.

But in Art. 196, the laws were shown to hold for $(a+x)^4$, and hence they also hold for $(a+x)^5$; and since they hold for $(a+x)^5$, they also hold for $(a+x)^6$; and so on.

Therefore the laws hold when the exponent is any positive integer, and equation (1) is proved for any positive integral value of n.

Note 1. The above method of proof is known as the *Method of Induction.*

Note 2. In place of the denominators $1\cdot 2$, $1\cdot 2\cdot 3$, etc., it is customary to write $\underline{|2}$, $\underline{|3}$, etc. The symbol $\underline{|n}$, read "*factorial n*," signifies the product of the natural numbers from 1 to n inclusive.

364. Putting $a = 1$ in equation (1), Art. 363, we obtain

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + \cdots$$

EXAMPLES.

365. Note. The Notes on page 164 apply with equal force to the examples in the present chapter. If the second term of the binomial is *negative*, it is convenient to enclose it, negative sign and all, in a parenthesis, before applying the laws of Art. 196. In reducing afterwards, care must be taken to apply the principles of Art. 192.

1. Expand $(m^{-\frac{1}{3}} - \sqrt{n})^5$.

$$(m^{-\frac{1}{3}} - \sqrt{n})^5 = [(m^{-\frac{1}{3}}) + (-n^{\frac{1}{2}})]^5.$$

The exponent of $(m^{-\frac{1}{3}})$ in the first term is 5, and decreases by 1 in each succeeding term.

The exponent of $(-n^{\frac{1}{2}})$ in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, 5; multiplying 5, the coefficient of the second term, by 4, the exponent of $(m^{-\frac{1}{3}})$ in that term, and dividing the result by the exponent of $(-n^{\frac{1}{2}})$ increased by 1, or 2, we have 10 as the coefficient of the third term; and so on. Hence,

$$\begin{aligned}
&[(m^{-\frac{1}{3}}) + (-n^{\frac{1}{2}})]^5 \\
&\quad = (m^{-\frac{1}{3}})^5 + 5(m^{-\frac{1}{3}})^4(-n^{\frac{1}{2}}) + 10(m^{-\frac{1}{3}})^3(-n^{\frac{1}{2}})^2 \\
&\qquad\qquad + 10(m^{-\frac{1}{3}})^2(-n^{\frac{1}{2}})^3 + 5(m^{-\frac{1}{3}})(-n^{\frac{1}{2}})^4 + (-n^{\frac{1}{2}})^5 \\
&\quad = m^{-\frac{5}{3}} - 5m^{-\frac{4}{3}}n^{\frac{1}{2}} + 10m^{-1}n - 10m^{-\frac{2}{3}}n^{\frac{3}{2}} \\
&\qquad\qquad + 5m^{-\frac{1}{3}}n^2 - n^{\frac{5}{2}}, \textit{ Ans.}
\end{aligned}$$

Expand the following:

2. $(c^{\frac{2}{3}} + d^{-\frac{3}{4}})^4$.

3. $(m^{-\frac{1}{2}} - n^2)^5$.

4. $\left(\frac{x}{y} - \frac{3y}{x}\right)^6$.

5. $(x^m + 2y^n)^5$.

6. $(a^3 + 3\sqrt{x})^4$.

7. $\left(\frac{m}{n} - \sqrt{mn}\right)^5$.

8. $\left(\frac{\sqrt[3]{x}}{\sqrt[3]{y^2}} + \frac{\sqrt[3]{y}}{\sqrt[3]{x^2}}\right)^3$.

9. $\left(m^2 - \frac{n^3}{2}\right)^4$.

10. $(a^{\frac{1}{2}} b^{-\frac{1}{3}} - a^{-\frac{1}{2}} b^{\frac{1}{3}})^5$.

11. $(\sqrt{a^3} - 3\sqrt[3]{a})^4$.

12. $\left(\frac{2x}{\sqrt{y}} + \frac{y}{2\sqrt{x}}\right)^4$.

13. $\left(a^{-2} - \frac{1}{3}x^{\frac{1}{2}}\right)^6$.

14. $(x^{\frac{3}{5}} + 3y^{-\frac{2}{5}})^5$.

15. $\left(\frac{a\sqrt{b}}{2x^{\frac{1}{2}}} - \frac{2\sqrt[3]{x}}{a^2 b^{\frac{1}{2}}}\right)^3$.

16. $(3a^{-\frac{3}{4}}\sqrt{b} - b^{-\frac{1}{2}}\sqrt[4]{a})^4$.

17. $\left(\sqrt{\frac{a}{b}} + 2\sqrt{\frac{b}{a}}\right)^6$.

Note. A *trinomial* may be raised to any power by the Binomial Theorem if two of its terms are enclosed in a parenthesis and regarded as a single term. (Compare Art. 195.)

Expand the following:

18. $(1 - x - x^2)^4$.

19. $(x^2 + x - 2)^4$.

20. $(1 + 2x - x^2)^4$.

21. $(1 - x + x^2)^5$.

366. *To find the rth or general term in the expansion of* $(a + x)^n$.

The following laws will be observed to hold for any term in the expansion of $(a + x)^n$, in equation (1), Art. 363:

1. The exponent of x is less by 1 than the number of the term.

2. The exponent of a is n minus the exponent of x.

3. The last factor of the numerator is greater by 1 than the exponent of a.

4. The last factor of the denominator is the same as the exponent of x.

Therefore, in the rth term, the exponent of x will be $r - 1$.
The exponent of a will be $n - (r - 1)$, or $n - r + 1$.
The last factor of the numerator will be $n - r + 2$.
The last factor of the denominator will be $r - 1$.

Hence, the rth term

$$=\frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)}a^{n-r+1}x^{r-1}.$$

EXAMPLES.

367. 1. Find the eighth term of $(3a^{\frac{1}{2}}-b^{-1})^{11}$.

$$(3a^{\frac{1}{2}}-b^{-1})^{11}=[(3a^{\frac{1}{2}})+(-b^{-1})]^{11}.$$

In this case, $r=8$, and $n=11$; hence the eighth term

$$=\frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a^{\frac{1}{2}})^4(-b^{-1})^7$$

$$=330(81a^2)(-b^{-7})=-26730a^2b^{-7}, \textit{ Ans.}$$

Note. If the second term of the binomial is negative, it should be enclosed, sign and all, in a parenthesis, before applying the formula.

Find the

2. Seventh term of $(a+x)^{11}$.

3. Sixth term of $(1+m)^{10}$.

4. Eighth term of $(c-d)^{12}$.

5. Fifth term of $(1-a^2)^{14}$..

6. Seventh term of $\left(\frac{a}{b}+\frac{b}{a}\right)^9$.

7. Fifth term of $(x-\sqrt{x})^{13}$.

8. Sixth term of $\left(a^{-3}-\frac{1}{2}ab\right)^9$.

9. Eighth term of $(x^{-1}+2y^{\frac{1}{2}})^{10}$.

10. Fourth term of $(a^{\frac{2}{3}}-3x^{-1})^{11}$.

11. Ninth term of $\left(\sqrt{m}+\frac{2}{\sqrt[4]{m}}\right)^{12}$.

XXXIV. THE THEOREM OF UNDETERMINED COEFFICIENTS.

368. A **Series** is a succession of terms so related that each may be derived from one or more of the others in accordance with some fixed law.

The simpler forms of series have already been exhibited in the progressions.

369. A *Finite Series* is one having a finite number of terms.

An *Infinite Series* is one the number of whose terms is unlimited.

The progressions in general are examples of finite series; but in Art. 351 we considered infinite geometrical series.

370. Infinite series may be developed by the process of Division, when the divisor is not exactly contained in the dividend.

Let it be required, for example, to divide 1 by $1-x$.

$$
\begin{array}{r|l}
1-x) & 1 \ (1+x+x^2+x^3+\cdots \\
& \underline{1-x} \\
& \quad\ \ x \\
& \quad\ \ \underline{x-x^2} \\
& \qquad\quad x^2 \\
& \qquad\quad \underline{x^2-x^3} \\
& \qquad\qquad\quad x^3
\end{array}
$$

Therefore, $$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$

Infinite series may also be obtained by the process of Evolution (see Examples 20 to 23, page 169), and by other methods, one of the most important of which will be considered in Art. 376.

371. A series is said to be *convergent* either when the sum of the first n terms approaches a certain fixed quantity as a limit (Art. 297), when n is indefinitely increased, or when the sum of all the terms is equal to a finite quantity.

A series is said to be *divergent* when the sum of the first n terms can be made to numerically exceed any assigned quantity, however great, by taking n sufficiently great.

372. Consider, for example, the infinite series

$$1 + x + x^2 + x^3 + \cdots$$

I. Suppose $x = x_1$, where x_1 is positive and < 1.

The sum of the first n terms is now

$$1 + x_1 + x_1^2 + x_1^3 + \cdots + x_1^{n-1} = \frac{1 - x_1^n}{1 - x_1} \qquad \text{(Art. 99)}$$

As n increases indefinitely, x_1^n decreases indefinitely, and approaches the limit 0.

Therefore the fraction $\frac{1 - x_1^n}{1 - x_1}$ approaches the limit $\frac{1}{1 - x_1}$.

That is, the sum of the first n terms approaches a certain fixed quantity as a limit, when n is indefinitely increased.

Hence the series is *convergent* when x is positive and < 1.

II. Suppose $x = 1$.

In this case, each term of the series is equal to 1, and the sum of the first n terms is equal to n; and this sum can be made to numerically exceed any assigned quantity however great, by taking n sufficiently great.

Hence the series is *divergent* when $x = 1$.

III. Suppose $x > 1$.

In this case, each term of the series after the first is > 1, and the sum of the first n terms is $> n$; and this sum can be made to numerically exceed any assigned quantity however great, by taking n sufficiently great.

Hence the series is *divergent* when x is > 1.

373. If an infinite series is *convergent*, the greater the number of terms taken, the more nearly does their sum approach to the value of the expression which produced the series; but if it is *divergent*, the sum diverges more and more from the value of the expression.

Consider, for example, the equation (Art. 370),

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$

Putting $x=.1$, in which case the series is convergent (Art. 372), the equation becomes

$$\frac{10}{9}=1+.1+.01+.001+\cdots$$

In this case, however great the number of terms taken, the sum can never be made exactly equal to $\frac{10}{9}$, but it approaches this value as a limit. (See Art. 352.)

Again, putting $x=10$, in which case the series is divergent, the equation becomes

$$-\frac{1}{9}=1+10+100+1000+\cdots$$

In this case it is evident that the sum of the terms diverges more and more from the value $-\frac{1}{9}$.

It follows from the above that an infinite series cannot be regarded as representing the value of the expression which produced it, unless it is convergent.

374. The infinite series

$$a+bx+cx^2+dx^3+\cdots$$

is convergent when $x=0$; for the sum of all the terms is equal to a when $x=0$.

THE THEOREM OF UNDETERMINED COEFFICIENTS.

375. An important method for expanding expressions into series is based on the following theorem, known as the *Theorem of Undetermined Coefficients.*

376. *If the series* $A+Bx+Cx^2+Dx^3+\cdots$ *is always equal to the series* $A'+B'x+C'x^2+D'x^3+\cdots$, *when* x *has any value which makes both series convergent, the coefficients of like powers of* x *in the two series will be equal; that is,* $A=A'$, $B=B'$, $C=C'$, *etc.*

For since the equation

$$A+Bx+Cx^2+Dx^3+\cdots=A'+B'x+C'x^2+D'x^3+\cdots$$

is satisfied when x has any value which makes both series convergent, and since both members are convergent when $x=0$ (Art. 374), it follows that the equation is satisfied when $x=0$.

Putting $x=0$, we have $A=A'$.

Subtracting A from the first member of the equation, and its equal A' from the second member, we obtain

$$Bx+Cx^2+Dx^3+\cdots=B'x+C'x^2+D'x^3+\cdots$$

Dividing through by x,

$$B+Cx+Dx^2+\cdots=B'+C'x+D'x^2+\cdots$$

This equation also is satisfied when x has any value which makes both members convergent; and putting $x=0$, we have

$$B=B'.$$

In like manner we may prove $C=C'$, $D=D'$, etc.

Note. The reason for limiting the theorem to values of x which make both series convergent, is that a convergent series evidently cannot be equal to a divergent series; and two divergent series cannot be equal, because two expressions neither of which is finite cannot be said to be equal.

377. Since a finite series is always convergent, it follows from the preceding article that if two finite series

$$A+Bx+Cx^2+\cdots+Kx^n \text{ and } A'+B'x+C'x^2+\cdots+K'x^n,$$

are equal for every value of x, the coefficients of like powers of x in the two series are equal.

APPLICATION TO THE EXPANSION OF FRACTIONS INTO SERIES.

378. 1. Expand $\dfrac{2-3x^2-x^3}{1-2x+3x^2}$ in ascending powers of x.

We have seen in Art. 370 that a fraction of the above form can be expanded into a series by dividing the numerator by the denominator; we therefore know that the proposed expansion is possible.

Assume then

$$\frac{2-3x^2-x^3}{1-2x+3x^2}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots, \qquad (1)$$

where A, B, C, D, E, ..., are quantities independent of x.

Clearing of fractions, and collecting the terms in the second member involving like powers of x, we have

$$2-3x^2-x^3=A+\left.\begin{array}{r}B\\-2A\end{array}\right|x+\left.\begin{array}{r}C\\-2B\\+3A\end{array}\right|x^2+\left.\begin{array}{r}D\\-2C\\+3B\end{array}\right|x^3+\left.\begin{array}{r}E\\-2D\\+3C\end{array}\right|x^4+\cdots \qquad (2)$$

The second member of (1) must express the value of the fraction for every value of x which makes the series convergent (Art. 373).

Hence equation (2) is satisfied when x has any value which makes both members convergent, and by the Theorem of Undetermined Coefficients, the coefficients of like powers of x in the two series must be equal; that is,

$$\begin{aligned}
A&=2.\\
B-2A&=0;\ \text{whence, } B=2A &&=4.\\
C-2B+3A&=-3;\ \text{whence, } C=2B-3A-3&&=-1.\\
D-2C+3B&=-1;\ \text{whence, } D=2C-3B-1&&=-15.\\
E-2D+3C&=0;\ \text{whence, } E=2D-3C&&=-27;\ \text{etc.}
\end{aligned}$$

Substituting these values in (1), we have

$$\frac{2-3x^2-x^3}{1-2x+3x^2}=2+4x-x^2-15x^3-27x^4-\cdots, \text{ \textit{Ans.}}$$

The result may be verified by division.

Note. A vertical line, called a *bar*, is often used instead of a parenthesis; thus,

$$\left.\begin{array}{r} +\ \ B \\ -2A \end{array}\right| x \text{ is equivalent to } (B-2A)x.$$

If the numerator and denominator contain only *even* powers of x, the expansion will involve only even powers of x; in this case the operation may be abridged by assuming a series containing only the even powers of x.

Thus, if the fraction were $\dfrac{2+4x^2-x^4}{1-3x^2+5x^4}$, we should assume it equal to $A+Bx^2+Cx^4+Dx^6+Ex^8+\cdots$

In like manner, if the numerator contains only *odd* powers of x, and the denominator only even powers, we should assume a series containing only the odd powers of x.

If every term of the numerator contains x, we may assume a series commencing with the lowest power of x in the numerator.

EXAMPLES.

Expand each of the following to five terms, in ascending powers of x:

2. $\dfrac{1-x}{1+x}$.

3. $\dfrac{2+5x}{1-3x}$.

4. $\dfrac{3-4x^2}{1+5x^2}$.

5. $\dfrac{2x}{3-2x^2}$.

6. $\dfrac{1-x-x^2}{1+x+x^2}$.

7. $\dfrac{x-3x^2-x^3}{1-2x-x^2}$.

8. $\dfrac{2-x+x^2}{1-x^2}$.

9. $\dfrac{1-2x^2}{1+2x-3x^2}$.

10. $\dfrac{2-3x+4x^2}{1+2x-5x^2}$.

11. $\dfrac{x^2+2x^3}{2-x-x^2}$.

12. $\dfrac{3+x-2x^3}{3-x^2+x^3}$.

13. $\dfrac{1-3x^2}{2-3x-2x^3}$.

If the lowest power of x in the denominator is higher than the lowest power in the numerator, we may determine by actual division what power of x will occur in the first term of the expansion; we should then assume the fraction equal to a series commencing with this power of x, the exponents of x in the succeeding terms increasing by unity as before.

14. Expand $\dfrac{1}{3x^2-x^3}$ in ascending powers of x.

Dividing 1 by $3x^2$, the quotient is $\dfrac{x^{-2}}{3}$; we then assume

$$\frac{1}{3x^2-x^3}=Ax^{-2}+Bx^{-1}+C+Dx+Ex^2+\cdots \qquad (1)$$

Clearing of fractions,

$$1=3A+\left.\begin{matrix}3B\\-A\end{matrix}\right|x+\left.\begin{matrix}3C\\-B\end{matrix}\right|x^2+\left.\begin{matrix}3D\\-C\end{matrix}\right|x^3+\left.\begin{matrix}3E\\-D\end{matrix}\right|x^4+\cdots$$

Equating the coefficients of like powers of x,

$$3A=1;$$
$$3B-A=0;$$
$$3C-B=0;$$
$$3D-C=0;$$
$$3E-D=0;\text{ etc.}$$

Whence, $A=\dfrac{1}{3}$, $B=\dfrac{1}{9}$, $C=\dfrac{1}{27}$, $D=\dfrac{1}{81}$, $E=\dfrac{1}{243}$, etc.

Substituting in (1), we have

$$\frac{1}{3x^2-x^3}=\frac{x^{-2}}{3}+\frac{x^{-1}}{9}+\frac{1}{27}+\frac{x}{81}+\frac{x^2}{243}+\cdots,\ \textit{Ans.}$$

Expand each of the following to five terms, in ascending powers of x:

15. $\dfrac{2}{3x^2-4x^3}$.

16. $\dfrac{1+x-x^2}{x-2x^2+3x^3}$.

17. $\dfrac{1-2x^2-x^3}{x^2+x^3-x^4}$.

18. $\dfrac{3-2x+x^3}{2x^3-x^4-2x^5}$.

APPLICATION TO THE EXPANSION OF RADICALS INTO SERIES.

379. 1. Expand $\sqrt{1-x}$ in ascending powers of x.

We have seen in Art. 204 that the square root of an imperfect square can be expanded into a series by the process of Evolution; we therefore know that the proposed expansion is possible. Assume then

$$\sqrt{1-x}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots \qquad (1)$$

Squaring both members, we have by Art. 194,

$$1-x=A^2+\left.\begin{array}{l}\\ 2AB\end{array}\right|x+\left.\begin{array}{l}B^2\\+2AC\end{array}\right|x^2+\left.\begin{array}{l}\\+2AD\\+2BC\end{array}\right|x^3+\left.\begin{array}{l}C^2\\+2AE\\+2BD\end{array}\right|x^4+\cdots$$

Equating the coefficients of like powers of x,

$$A^2=1;\ \text{whence, } A=1.$$

$$2AB=-1;\ \text{whence, } B=-\frac{1}{2A}=-\frac{1}{2}.$$

$$B^2+2AC=0;\ \text{whence, } C=-\frac{B^2}{2A}=-\frac{1}{8}.$$

$$2AD+2BC=0;\ \text{whence, } D=-\frac{BC}{A}=-\frac{1}{16}.$$

$$C^2+2AE+2BD=0;\ \text{whence, } E=-\frac{C^2+2BD}{2A}=-\frac{5}{128};$$

etc.

Substituting these values in (1), we have

$$\sqrt{1-x}=1-\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}-\frac{5x^4}{128}-\cdots,\ \textit{Ans.}$$

The result may be verified by the method of Art. 204.

Note. The equation $A^2=1$ gives $A=\pm 1$; and taking the *negative* value of A, we should find $B=\frac{1}{2}$, $C=\frac{1}{8}$, $D=\frac{1}{16}$, etc.

Thus another answer to the example is

$$-1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{16}+\cdots$$

EXAMPLES.

Expand each of the following to five terms, in ascending powers of x:

2. $\sqrt{1+2x}$.   4. $\sqrt{1-2x+3x^2}$.   6. $\sqrt[3]{1-x}$.

3. $\sqrt{1-3x}$.   5. $\sqrt{1+x-x^2}$.   7. $\sqrt[3]{1+x+x^2}$.

APPLICATION TO THE DECOMPOSITION OF RATIONAL FRACTIONS.

380. If the denominator of a fraction can be resolved into factors, each of the first degree in x, and the numerator is of a lower degree than the denominator, the Theorem of Undetermined Coefficients enables us to express the given fraction as the sum of two or more *partial fractions*, whose denominators are factors of the given denominator, and whose numerators are independent of x.

Case I.

381. *When no two factors of the denominator are equal.*

1. Separate $\dfrac{19x+1}{(3x-1)(5x+2)}$ into partial fractions.

Assume
$$\frac{19x+1}{(3x-1)(5x+2)}=\frac{A}{3x-1}+\frac{B}{5x+2}, \qquad (1)$$

where A and B are quantities independent of x.

Clearing of fractions, we have

$$\begin{aligned}19x+1&=A(5x+2)+B(3x-1)\\&=(5A+3B)x+2A-B. \qquad (2)\end{aligned}$$

The second member of equation (1) must express the value of the given fraction for every value of x.

Hence equation (2) is satisfied by every value of x, and by Art. 377 the coefficients of like powers of x in the two members are equal.

That is, $5A + 3B = 19$,

and $2A - B = 1$.

Solving these equations, we obtain $A = 2$ and $B = 3$.

Substituting in (1), we have

$$\frac{19x + 1}{(3x-1)(5x+2)} = \frac{2}{3x-1} + \frac{3}{5x+2}, \quad Ans.$$

The result may be verified by adding the partial fractions.

2. Separate $\dfrac{x+4}{2x - x^2 - x^3}$ into partial fractions.

The factors of $2x - x^2 - x^3$ are x, $1-x$, and $2+x$ (Art. 283). Assume then

$$\frac{x+4}{2x - x^2 - x^3} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{2+x}. \qquad (1)$$

Clearing of fractions, we have

$$x + 4 = A(1-x)(2+x) + Bx(2+x) + Cx(1-x).$$

This equation, being satisfied by every value of x, is satisfied when $x = 0$.

Putting $x = 0$, we have $4 = 2A$, or $A = 2$.

Again, the equation is satisfied when $x = 1$.

Putting $x = 1$, we have $5 = 3B$, or $B = \dfrac{5}{3}$.

The equation is also satisfied when $x = -2$.

Putting $x = -2$, we have $2 = -6C$, or $C = -\dfrac{1}{3}$.

Substituting in (1), we obtain

$$\frac{x+4}{2x - x^2 - x^3} = \frac{2}{x} + \frac{\frac{5}{3}}{1-x} + \frac{-\frac{1}{3}}{2+x}$$

$$= \frac{2}{x} + \frac{5}{3(1-x)} - \frac{1}{3(2+x)}, \quad Ans.$$

Note. The student should compare the above method of finding A and B with that used in Example 1.

EXAMPLES.

Separate the following into partial fractions:

3. $\dfrac{14x-25}{4x^2-25}$.

4. $\dfrac{4x+15}{3x^2+5x}$.

5. $\dfrac{x^2-45}{2x^3-18x}$.

6. $\dfrac{13x+10}{6x^3-13x^2-5x}$.

7. $\dfrac{ax-14a^2}{x^2-3ax-4a^2}$.

8. $\dfrac{7x+9}{9+9x-4x^2}$.

9. $\dfrac{2x^2-17x-24}{(x+1)(4x^2-9)}$.

10. $\dfrac{2x^2-20}{(x^2-4)(x^2-1)}$.

11. $\dfrac{4x-14}{4x^2-20x+23}$.

Case II.

382. *When all the factors of the denominator are equal.*

Example. Separate $\dfrac{x^2-11x+26}{(x-3)^3}$ into partial fractions.

If we attempt to solve the example by the method of Case I., we should assume

$$\frac{x^2-11x+26}{(x-3)^3}=\frac{A}{x-3}+\frac{B}{x-3}+\frac{C}{x-3}.$$

That is, $\dfrac{x^2-11x+26}{(x-3)^3}=\dfrac{A+B+C}{x-3}.$

But this is evidently impossible, for the given fraction cannot be reduced to an equivalent fraction having $x-3$ for a denominator, and a numerator independent of x.

Let us now substitute in the given fraction $y+3$ in place of x; we then have

$$\frac{(y+3)^2-11(y+3)+26}{y^3}=\frac{y^2-5y+2}{y^3}=\frac{1}{y}-\frac{5}{y^2}+\frac{2}{y^3}.$$

Replacing y by $x-3$, the result takes the form

$$\frac{1}{x-3}-\frac{5}{(x-3)^2}+\frac{2}{(x-3)^3}.$$

This shows that the given fraction can be expressed as the sum of three partial fractions, whose numerators are independent of x, and whose denominators are the powers of $x-3$ beginning with the first and ending with the third.

A similar result will hold in any example under Case II.; the number of partial fractions being equal to the number of equal factors in the denominator of the given fraction.

EXAMPLES.

383. 1. Separate $\dfrac{6x+5}{(3x+5)^2}$ into partial fractions.

In accordance with the principle stated in Art. 382, we assume the given fraction equal to the sum of *two* partial fractions, whose denominators are the powers of $3x+5$ beginning with the first and ending with the *second;* that is,

$$\frac{6x+5}{(3x+5)^2}=\frac{A}{3x+5}+\frac{B}{(3x+5)^2}.$$

Clearing of fractions, we have

$$6x+5=A(3x+5)+B$$
$$=3Ax+5A+B.$$

Equating the coefficients of like powers of x,

$$3A=6,$$

and
$$5A+B=5.$$

Solving these equations, we have $A=2$ and $B=-5$.

Whence, $\dfrac{6x+5}{(3x+5)^2}=\dfrac{2}{3x+5}-\dfrac{5}{(3x+5)^2}$, *Ans.*

Separate the following into partial fractions:

2. $\dfrac{2x-13}{x^2+10x+25}$.

3. $\dfrac{x^2}{(x-2)^3}$.

4. $\dfrac{3x^2-4}{(x+1)^3}$.

5. $\dfrac{18x^2+12x-3}{(3x+2)^3}$.

6. $\dfrac{x(5x-4)}{(5x-2)^3}$.

7. $\dfrac{x(x+2)^2}{(x+1)^4}$.

8. $\dfrac{2x^3-10x^2+17x-10}{(x-1)^4}$.

9. $\dfrac{4x^3-18x^2}{(2x-3)^4}$.

CASE III.

384. *When some of the factors of the denominator are equal.*

1. Separate $\frac{3x+2}{x(x+1)^3}$ into partial fractions.

The method in Case III. is a combination of the methods of Cases I. and II.; we assume

$$\frac{3x+2}{x(x+1)^3}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}. \qquad (1)$$

Clearing of fractions,

$$\begin{aligned}3x+2&=A(x+1)^3+Bx(x+1)^2+Cx(x+1)+Dx\\&=(A+B)x^3+(3A+2B+C)x^2\\&\quad+(3A+B+C+D)x+A.\end{aligned}$$

Equating the coefficients of like powers of x,

$$A+B=0,$$
$$3A+2B+C=0,$$
$$3A+B+C+D=3,$$

and $$A=2.$$

Solving these equations, we have

$$A=2,\ B=-2,\ C=-2,\text{ and } D=1.$$

Substituting in (1),

$$\frac{3x+2}{x(x+1)^3}=\frac{2}{x}-\frac{2}{x+1}-\frac{2}{(x+1)^2}+\frac{1}{(x+1)^3}, \quad \textit{Ans.}$$

Note. It is impracticable to give an illustrative example for every possible case; but the student should find no difficulty in assuming the proper partial fractions if attention is given to the following general rule:

A fraction of the form $\frac{X}{(x+a)(x+b)\cdots(x+m)^r\cdots}$ should be put equal to

$$\frac{A}{x+a}+\frac{B}{x+b}+\cdots+\frac{E}{x+m}+\frac{F}{(x+m)^2}+\cdots+\frac{K}{(x+m)^r}+\cdots$$

Single factors like $x+a$ and $x+b$ having single partial fractions corresponding, arranged as in Case I.; and repeated factors like $(x+m)^r$ having r partial fractions corresponding, arranged as in Case II.

EXAMPLES.

Separate the following into partial fractions:

2. $\dfrac{8-3x-x^2}{x(x+2)^2}$.

3. $\dfrac{3x-1}{x^2(x+1)^2}$.

4. $\dfrac{3x^2-7x+3}{(2x-3)(2x^2-7x+6)}$.

5. $\dfrac{3x^3-11x^2+13x-4}{x(x-1)(x-2)^2}$.

6. $\dfrac{15-7x+3x^2-3x^3}{x^4+5x^3}$.

7. $\dfrac{5x^2+3x+2}{x^3(x+1)^2}$.

385. If the degree of the numerator is equal to, or greater than, that of the denominator, the preceding methods are inapplicable.

Thus, let it be required to separate $\dfrac{x^3-3x^2-1}{x^2-x}$ into partial fractions.

If we proceed as in Case I., we should assume

$$\frac{x^3-3x^2-1}{x^2-x}=\frac{A}{x}+\frac{B}{x-1}.$$

Clearing of fractions and uniting terms,

$$x^3-3x^2-1=(A+B)x-A.$$

Equating the coefficients of x^3, we have $1=0$, a result which shows that the method of Case I. is inapplicable.

But by actual division, we obtain

$$\frac{x^3-3x^2-1}{x^2-x}=x-2+\frac{-2x-1}{x^2-x}. \quad (1)$$

We can now separate $\dfrac{-2x-1}{x^2-x}$ into partial fractions by the method of Case I.; the result is

$$\frac{1}{x}-\frac{3}{x-1}.$$

Substituting in (1), we have

$$\frac{x^3-3x^2-1}{x^2-x}=x-2+\frac{1}{x}-\frac{3}{x-1}, \quad \textit{Ans.}$$

EXAMPLES.

Separate the following into entire quantities and partial fractions:

1. $\dfrac{8x^3 - 36x^2 - 2}{(2x-5)(2x+1)}$.

2. $\dfrac{3x^3 + 19x^2 + 35x}{(x+2)^3}$.

3. $\dfrac{5x^6 + 5x^5 - 2x^2 + 3}{x^4 + x^3}$.

4. $\dfrac{3x^5 - 2x^4 + 22x^2 + 9x}{(x^2-1)^2}$.

5. $\dfrac{2x^6 - 2x^5 - 7x^4 + 2x^2 + x - 1}{x^4 - x^3}$.

APPLICATION TO THE REVERSION OF SERIES.

386. Note. To *revert* a given series $y = a + bx^m + cx^n + \cdots$ is to express x in terms of y.

Example. Revert the series

$$y = 2x + x^2 - 2x^3 - 3x^4 + \cdots$$

Assume $x = Ay + By^2 + Cy^3 + Dy^4 + \cdots$ (1)

Substituting in this the given value of y, we have

$$\begin{aligned} x = {} & A(2x + x^2 - 2x^3 - 3x^4 + \cdots) \\ & + B(4x^2 + x^4 + 4x^3 - 8x^4 + \cdots) \\ & + C(8x^3 + 12x^4 + \cdots) \\ & + D(16x^4 + \cdots) + \cdots \end{aligned}$$

That is, $x = 2Ax + \begin{array}{r|r|r|l} A & x^2 - 2A & x^3 - 3A & x^4 + \cdots \\ +4B & +4B & -7B & \\ & +8C & +12C & \\ & & +16D & \end{array}$

Equating the coefficients of like powers of x,

$$2A = 1;$$
$$A + 4B = 0;$$
$$-2A + 4B + 8C = 0;$$
$$-3A - 7B + 12C + 16D = 0; \text{ etc.}$$

Solving these equations,

$$A=\frac{1}{2},\ B=-\frac{1}{8},\ C=\frac{3}{16},\ D=-\frac{13}{128},\ \text{etc.}$$

Substituting in (1), we have

$$x=\frac{1}{2}y-\frac{1}{8}y^2+\frac{3}{16}y^3-\frac{13}{128}y^4+\cdots,\ \textit{Ans.}$$

If the even powers of x are wanting in the given series, the operation may be abridged by assuming x equal to a series containing only the *odd* powers of y.

Thus, to revert the series $y=x-x^3+x^5-x^7+\cdots$, we should assume

$$x=Ay+By^3+Cy^5+Dy^7+\cdots$$

If the *odd* powers of x are wanting in the given series, the reversion of the series cannot be effected by the method previously given. But by substituting another letter, say t, for x^2, we may revert the series and express t in terms of y; and by taking the square root of the result, x itself may be expressed in terms of y.

If the first term of the given series is independent of x, it is impossible, by the method previously given, to express x definitely in terms of y; but it is possible to express it in the form of a series in which y is the only unknown quantity.

Let it be required, for example, to revert the series

$$y=2+2x+x^2-2x^3-3x^4+\cdots$$

The series may be written

$$y-2=2x+x^2-2x^3-3x^4+\cdots$$

We then assume

$$x=A(y-2)+B(y-2)^2+C(y-2)^3+D(y-2)^4+\cdots$$

Proceeding as in Ex. 1, we find

$$x=\frac{1}{2}(y-2)-\frac{1}{8}(y-2)^2+\frac{3}{16}(y-2)^3-\frac{13}{128}(y-2)^4+\cdots$$

EXAMPLES.

387. Revert each of the following to four terms:

1. $y = x + x^2 + x^3 + x^4 + \cdots$

2. $y = 3x - 2x^2 + 3x^3 - 4x^4 + \cdots$

3. $y = \frac{x}{2} - \frac{3x^2}{4} + \frac{5x^3}{6} - \frac{7x^4}{8} + \cdots$

4. $y = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \cdots$

5. $y = x - x^3 + x^5 - x^7 + \cdots$

6. $y = \frac{x}{2} - \frac{x^2}{3} + \frac{x^3}{4} - \frac{x^4}{5} + \cdots$

7. $y = 3x + 5x^2 + 7x^3 + 11x^4 + \cdots$

8. $y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

XXXV. THE BINOMIAL THEOREM.

FRACTIONAL AND NEGATIVE EXPONENTS.

388. It was proved in Art. 364 that when n is a positive integer,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}}x^2 + \frac{n(n-1)(n-2)}{\underline{|3}}x^3 + \cdots \quad (1)$$

Proof of the Theorem for any Exponent.

389. I. *When the exponent is a positive fraction.*

Let the exponent be $\frac{p}{q}$, p and q being positive integers.

Then, $\quad (1+x)^{\frac{p}{q}} = \sqrt[q]{(1+x)^p}$ (Art. 218)

$= \sqrt[q]{1+px+\cdots}$, by (1).

It is evident that a process may be found, analogous to those of Arts. 203 and 208, for expanding $\sqrt[q]{1+px+\cdots}$ in ascending powers of x; and the first term of the result will evidently be 1. Assume then,

$$\sqrt[q]{1+px+\cdots} = 1 + Mx + Nx^2 + \cdots \quad (2)$$

Raising both members to the qth power, we have

$$1+px+\cdots = [1+(Mx+Nx^2+\cdots)]^q$$
$$= 1+q(Mx+Nx^2+\cdots)+\cdots, \text{ by (1).}$$

This equation being satisfied by every value of x which makes both members convergent, by the Theorem of Undetermined Coefficients (Art. 376) the coefficients of x in the two series are equal.

That is, $\quad p = qM$, or $M = \frac{p}{q}$.

Substituting this value in (2), we have

$$(1+x)^{\frac{p}{q}} = 1 + \frac{p}{q}x + \cdots \quad (3)$$

II. *When the exponent is a negative quantity.*

Let the exponent be $-s$, s being a positive quantity.

Then, $$(1+x)^{-s}=\frac{1}{(1+x)^s} \text{ (Art. 221)}$$

$$=\frac{1}{1+sx+\cdots}, \text{ by (1) or (3).}$$

Whence by actual division, we obtain

$$(1+x)^{-s}=1-sx+\cdots \tag{4}$$

From (1), (3), and (4), we observe that whether n is positive or negative, integral or fractional, the form of the expansion is

$$(1+x)^n=1+nx+Ax^2+Bx^3+\cdots \tag{5}$$

Writing $\frac{x}{a}$ in place of x, we obtain

$$\left(1+\frac{x}{a}\right)^n=1+n\frac{x}{a}+A\frac{x^2}{a^2}+B\frac{x^3}{a^3}+\cdots$$

Multiplying both members by a^n, we have

$$(a+x)^n=a^n+na^{n-1}x+Aa^{n-2}x^2+Ba^{n-3}x^3+\cdots \tag{6}$$

This result is in accordance with the *second*, *third*, and *fourth* laws of Art. 196; hence these three laws hold for any value of the exponent.

390. We will now prove the *fifth* law of Art. 196 for any value of the exponent.

Let P and Q denote the coefficients of x^r and x^{r+1} in the second member of (5); then (5) and (6) may be written

$$(1+x)^n=1+nx+\cdots+Px^r+Qx^{r+1}+\cdots, \tag{7}$$

and

$$(a+x)^n=a^n+na^{n-1}x+\cdots+Pa^{n-r}x^r+Qa^{n-r-1}x^{r+1}+\cdots \tag{8}$$

In (8) put $a=1+y$, and $x=z$; then,

$$(1+y+z)^n=(1+y)^n+\cdots+P(1+y)^{n-r}z^r+\cdots \tag{9}$$

Again, in (7) put $x = z + y$; then,

$$(1 + z + y)^n = 1 + \cdots + P(z + y)^r + Q(z + y)^{r+1} + \cdots$$

Expanding the powers of $z + y$ by aid of (8), we have

$$(1 + z + y)^n = 1 + \cdots + P[z^r + rz^{r-1}y + \cdots] \\ + Q[z^{r+1} + (r+1)z^r y + \cdots] + \cdots \quad (10)$$

The first members of (9) and (10) being identical, their second members are equal for every value of z which makes both series convergent; and by the Theorem of Undetermined Coefficients, the coefficients of z^r in the two series are equal; that is,

$$P(1 + y)^{n-r} = P + Q(r + 1)y + \text{terms in } y^2,\ y^3, \text{ etc.}$$

Expanding the first member by aid of (7), this becomes

$$P[1 + (n - r)y + \cdots] = P + Q(r + 1)y + \cdots$$

This equation being satisfied by every value of y which makes both members convergent, the coefficients of y in the two series are equal.

Therefore, $P(n - r) = Q(r + 1)$, or $Q = \dfrac{P(n - r)}{r + 1}$.

That is, the coefficient Q is equal to the coefficient of the preceding term in (8), multiplied by the exponent of a in that term, and divided by the exponent of x increased by 1.

Thus the fifth law of Art. 196 is proved to hold for any value of the exponent.

391. By aid of the law proved in Art. 390, the coefficients of the terms after the second in the second member of (8), Art. 390, may be readily found as in (1), Art. 363.

Thus, $$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{\underline{|2}}a^{n-2}x^2 \\ + \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}x^3 + \cdots,$$

and the Binomial Theorem is proved in its most general form.

If n is a positive integer, the number of terms in the series is $n+1$; for all coefficients after the $(n+1)$st contain the factor $n-n$, or 0. (Compare Art. 196.)

But if n is fractional or negative, the expansion never terminates, since no one of the quantities $n-1$, $n-2$, ..., can become equal to zero. The development in this case furnishes an infinite series, which however expresses the value of $(a+x)^n$ only for such values of a and x as make the series convergent. (Compare Art. 373.)

EXAMPLES.

392. In expanding expressions by the Binomial Theorem when the exponent is fractional or negative, it is convenient to obtain the exponents and coefficients of the terms by aid of the laws of Art. 196, which have been proved to hold universally.

If the second term is negative, it should be enclosed, sign and all, in a parenthesis, as in Arts. 365 and 367, before applying the laws.

1. Expand $(a+x)^{\frac{2}{3}}$ to four terms.

The exponent of a in the first term is $\frac{2}{3}$, and decreases by 1 in each succeeding term.

The exponent of x in the second term is 1, and increases by 1 in each succeeding term.

The coefficient of the first term is 1; of the second term, $\frac{2}{3}$; multiplying $\frac{2}{3}$, the coefficient of the second term, by $-\frac{1}{3}$, the exponent of a in that term, and dividing the product, $-\frac{2}{9}$, by the exponent of x increased by 1, or 2, we have $-\frac{1}{9}$ as the coefficient of the third term; and so on. Hence,

$$(a+x)^{\frac{2}{3}} = a^{\frac{2}{3}} + \frac{2}{3}a^{-\frac{1}{3}}x - \frac{1}{9}a^{-\frac{4}{3}}x^2 + \frac{4}{81}a^{-\frac{7}{3}}x^3 - \cdots. \textit{ Ans.}$$

2. Expand $(1-2x^{-\frac{1}{2}})^{-2}$ to five terms.

$$
\begin{aligned}
(1-2x^{-\frac{1}{2}})^{-2} &= [1+(-2x^{-\frac{1}{2}})]^{-2}\\
&= 1^{-2}-2\cdot 1^{-3}\cdot(-2x^{-\frac{1}{2}})+3\cdot 1^{-4}\cdot(-2x^{-\frac{1}{2}})^2\\
&\quad -4\cdot 1^{-5}\cdot(-2x^{-\frac{1}{2}})^3+5\cdot 1^{-6}\cdot(-2x^{-\frac{1}{2}})^4-\cdots\\
&= 1+4x^{-\frac{1}{2}}+12x^{-1}+32x^{-\frac{3}{2}}+80x^{-2}+\cdots, \textit{ Ans.}
\end{aligned}
$$

3. Expand $\dfrac{1}{\sqrt[3]{a^{-1}+3x^{\frac{1}{2}}}}$ to four terms.

$$
\begin{aligned}
\frac{1}{\sqrt[3]{a^{-1}+3x^{\frac{1}{2}}}} &= \frac{1}{(a^{-1}+3x^{\frac{1}{2}})^{\frac{1}{3}}} = [(a^{-1})+(3x^{\frac{1}{2}})]^{-\frac{1}{3}}\\
&= (a^{-1})^{-\frac{1}{3}}-\frac{1}{3}(a^{-1})^{-\frac{4}{3}}(3x^{\frac{1}{2}})+\frac{2}{9}(a^{-1})^{-\frac{7}{3}}(3x^{\frac{1}{2}})^2\\
&\quad -\frac{14}{81}(a^{-1})^{-\frac{10}{3}}(3x^{\frac{1}{2}})^3+\cdots\\
&= a^{\frac{1}{3}}-a^{\frac{4}{3}}x^{\frac{1}{2}}+2a^{\frac{7}{3}}x-\frac{14}{3}a^{\frac{10}{3}}x^{\frac{3}{2}}+\cdots, \textit{ Ans.}
\end{aligned}
$$

Expand each of the following to five terms:

4. $(a+x)^{\frac{5}{2}}$.

5. $(1+x)^{-6}$.

6. $(1-x)^{-\frac{3}{5}}$.

7. $\sqrt{a-x}$.

8. $\dfrac{1}{\sqrt[3]{1+x}}$.

9. $\dfrac{1}{(a-x)^3}$.

10. $\dfrac{1}{c^{\frac{3}{2}}+d}$.

11. $(x^{-\frac{1}{2}}-3y)^{\frac{2}{3}}$.

12. $(a-2x^2)^{-\frac{1}{4}}$.

13. $\left(a^{\frac{5}{6}}+\dfrac{b^{-1}}{2}\right)^{-2}$.

14. $(x^4+4ab)^{\frac{3}{4}}$.

15. $\left(1+\dfrac{6x}{y}\right)^{-\frac{5}{3}}$.

16. $\dfrac{1}{(a^{-1}-3y^{-2})^4}$.

17. $(4a^2+x^{-\frac{2}{3}})^{\frac{7}{2}}$.

18. $\left(\dfrac{1}{\sqrt[3]{m^2}}-2\sqrt{n^3}\right)^{-\frac{3}{2}}$.

393. The formula for the rth term of $(a+x)^n$ (Art. 366) holds for fractional and negative values of n, since it was derived from an expansion which has been proved to hold universally.

EXAMPLES.

1. Find the seventh term of $(a-3x^{-\frac{3}{2}})^{-\frac{1}{3}}$.

$$(a-3x^{-\frac{3}{2}})^{-\frac{1}{3}}=[a+(-3x^{-\frac{3}{2}})]^{-\frac{1}{3}}.$$

In this case $r=7$, and $n=-\frac{1}{3}$; hence the seventh term

$$=\frac{-\frac{1}{3}\cdot-\frac{4}{3}\cdot-\frac{7}{3}\cdot-\frac{10}{3}\cdot-\frac{13}{3}\cdot-\frac{16}{3}}{1\cdot2\cdot3\cdot4\cdot5\cdot6}a^{-\frac{19}{3}}(-3x^{-\frac{3}{2}})^{6}$$

$$=\frac{728}{3^8}a^{-\frac{19}{3}}(3^6x^{-9})=\frac{728}{9}a^{-\frac{19}{3}}x^{-9},\ \textit{Ans.}$$

Find the

2. Eighth term of $(a+x)^{\frac{1}{2}}$.

3. Twelfth term of $(1+m)^{-4}$.

4. Fifth term of $(1-a^2)^{-\frac{3}{2}}$.

5. Seventh term of $(a-x)^{\frac{5}{2}}$.

6. Sixth term of $(a^4+b^{\frac{1}{2}})^{-\frac{7}{2}}$.

7. Seventh term of $(x^{-1}-y^{-\frac{1}{2}})^{\frac{4}{3}}$.

8. Sixth term of $\frac{1}{\sqrt[4]{x^2+y^3}}$.

9. Eleventh term of $(a^{\frac{1}{2}}+2x)^{\frac{7}{2}}$.

10. Ninth term of $\frac{1}{(n^{-\frac{2}{3}}-c^{-2})^7}$.

11. Sixth term of $(a^{\frac{2}{3}}+3x^{-1})^{-\frac{2}{3}}$.

12. Eighth term of $\left(x^3y-\frac{1}{\sqrt[3]{z^2}}\right)^{-5}$.

394. *To find any root of a number approximately by the Binomial Theorem.*

1. Find the approximate value of $\sqrt[3]{25}$ to five places of decimals.

$$\sqrt[3]{25} = 25^{\frac{1}{3}} = (27-2)^{\frac{1}{3}} = (3^3-2)^{\frac{1}{3}}.$$

Expanding by the Binomial Theorem, we have

$$[(3^3)+(-2)]^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} + \frac{1}{3}(3^3)^{-\frac{2}{3}}(-2) - \frac{1}{9}(3^3)^{-\frac{5}{3}}(-2)^2$$
$$+\frac{5}{81}(3^3)^{-\frac{8}{3}}(-2)^3 - \cdots$$
$$= 3 - \frac{2}{3\cdot 3^2} - \frac{4}{9\cdot 3^5} - \frac{40}{81\cdot 3^8} - \cdots$$

Expressing the value of each fraction approximately to five places of decimals, we have

$$\sqrt[3]{25} = 3 - .07407 - .00183 - .00008 - \cdots$$
$$= 2.92402, \text{ \textit{Ans.}}$$

RULE.

Separate the given number into two parts, the first of which is the nearest perfect power of the same degree as the required root.

Expand the result by the Binomial Theorem.

Note. If the second term of the binomial is small compared with the first, the terms of the expansion diminish rapidly; but if the second term is large compared with the first, it requires a great many terms to ensure any degree of accuracy.

EXAMPLES.

Find the approximate values of the following to five places of decimals:

2. $\sqrt{10}$. **4.** $\sqrt[3]{9}$. **6.** $\sqrt[4]{17}$.

3. $\sqrt{99}$. **5.** $\sqrt[4]{78}$. **7.** $\sqrt[5]{28}$.

XXXVI. LOGARITHMS.

395. Every positive number may be expressed, exactly or approximately, as a power of 10; thus,

$$100 = 10^2; \ 13 = 10^{1.1139\cdots}; \text{ etc.}$$

When thus expressed, the corresponding exponent is called its **Logarithm to the base 10**; thus, 2 is the logarithm of 100 to the base 10, a relation which is written

$$\log_{10} 100 = 2, \text{ or simply } \log 100 = 2.$$

And in general, if $10^x = m$, then $x = \log m$.

396. Any positive number except unity may be taken as the base of a system of logarithms; thus, if $a^x = m$, then $x = \log_a m$.

Logarithms to the base 10 are called *Common Logarithms*, and are the only ones used for numerical computations.

If no base is expressed, the base 10 is understood.

397. By Arts. 220 and 221, we have

$$10^0 = 1, \qquad 10^{-1} = \frac{1}{10} = .1,$$

$$10^1 = 10, \qquad 10^{-2} = \frac{1}{100} = .01,$$

$$10^2 = 100, \qquad 10^{-3} = \frac{1}{1000} = .001, \text{ etc.}$$

Whence, by the definition of Art. 395,

$$\log 1 = 0, \qquad \log .1 = -1 = 9 - 10,$$

$$\log 10 = 1, \qquad \log .01 = -2 = 8 - 10,$$

$$\log 100 = 2, \qquad \log .001 = -3 = 7 - 10, \text{ etc.}$$

Note. The second form of the results for log .1, log .01, etc., is preferable in practice. In each of the last six equations the base 10 is understood (Art. 396).

398. It is evident from Art. 397 that the logarithm of a number greater than 1 is positive, and that the logarithm of a number between 0 and 1 is negative.

399. If a number is not an exact power of 10, its common logarithm can only be expressed approximately; the integral part of the logarithm is called the *characteristic*, and the decimal part the *mantissa*.

For example, $\log 13 = 1.1139$.

In this case the characteristic of the logarithm is 1, and the mantissa is .1139.

400. It is evident from the first column of Art. 397 that the logarithm of any number between

1 and 10 is equal to 0 plus a decimal;
10 and 100 is equal to 1 plus a decimal;
100 and 1000 is equal to 2 plus a decimal; etc.

Hence, the characteristic of the logarithm of a number with *one* figure to the left of its decimal point, is 0; with *two* figures to the left of the decimal point, is 1; with *three* figures to the left of the decimal point, is 2; etc.

401. In like manner, from the second column of Art. 397, the logarithm of a decimal between

1 and .1 is equal to 9 plus a decimal -10;
.1 and .01 is equal to 8 plus a decimal -10;
.01 and .001 is equal to 7 plus a decimal -10; etc.

Hence, the characteristic of the logarithm of a decimal with *no* ciphers between its decimal point and first significant figure, is 9, with -10 after the mantissa; of a decimal with *one* cipher between its point and first figure is 8, with -10 after the mantissa; of a decimal with *two* ciphers between its point and first figure, is 7, with -10 after the mantissa; etc.

402. For reasons which will be given hereafter, only the mantissa of the logarithm is given in a table of logarithms of numbers; the characteristic must be supplied by the reader.

The rules for characteristic are based on Arts. 400 and 401:

I. *If the number is greater than* 1, *the characteristic is* 1 *less than the number of places to the left of the decimal point.*

II. *If the number is less than* 1, *subtract the number of ciphers between the decimal point and first significant figure from* 9, *writing* -10 *after the mantissa.*

Thus, characteristic of $\log 906328.5 = 5$;

characteristic of $\log\ .007023 = 7$, with -10 after the mantissa.

Note. Some writers, in dealing with the characteristics of negative logarithms, combine the two portions of the characteristic, and write the result as a *negative characteristic* before the mantissa.

Thus, instead of $7.6036 - 10$, the student will frequently find $\bar{3}.6036$, a minus sign being written over the characteristic to denote that it alone is negative, the mantissa being always positive.

PROPERTIES OF LOGARITHMS.

403. *In any system, the logarithm of unity is zero.*

For since $a^0 = 1$, we have $\log_a 1 = 0$ (Art. 395).

404. *In any system, the logarithm of the base itself is unity.*

For since $a^1 = a$, we have $\log_a a = 1$.

405. *In any system whose base is greater than unity, the logarithm of zero is minus infinity.*

For if a is >1, we have $a^{-\infty} = \frac{1}{a^\infty} = \frac{1}{\infty} = 0$ (Art. 300).

Whence by Art. 395, $\log_a 0 = -\infty$.

Note. As stated in Art. 301, no literal meaning can be attached to the result $\log_a 0 = -\infty$; it must be interpreted as indicated in Art. 300.

That is, if in any system whose base is greater than unity, a number approaches zero as a limit, its logarithm is negative, and increases without limit in absolute value.

406. *In any system, the logarithm of a product is equal to the sum of the logarithms of its factors.*

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ a^y = n \end{array}\right\} \text{; whence, by Art. 395, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_a n. \end{array}\right.$$

Multiplying, we have

$$a^x \times a^y = mn, \text{ or } a^{x+y} = mn.$$

Whence, $\log_a mn = x + y.$

Substituting the values of x and y, we have

$$\log_a mn = \log_a m + \log_a n.$$

In like manner, the theorem may be proved for the product of three or more factors.

407. By aid of the theorem of Art. 406, the logarithm of any composite number may be found when the logarithms of its factors are known.

1. Given $\log 2 = .3010$, and $\log 3 = .4771$; find $\log 72$.

$$\begin{aligned} \log 72 &= \log(2 \times 2 \times 2 \times 3 \times 3) \\ &= \log 2 + \log 2 + \log 2 + \log 3 + \log 3 \\ &= 3 \times \log 2 + 2 \times \log 3 \\ &= .9030 + .9542 \\ &= 1.8572, \textit{ Ans.} \end{aligned}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, $\log 5 = .6990$, and $\log 7 = .8451$; find:

2. $\log 21$.	**7.** $\log 98$.	**12.** $\log 135$.	**17.** $\log 1134$.
3. $\log 63$.	**8.** $\log 105$.	**13.** $\log 168$.	**18.** $\log 5145$.
4. $\log 56$.	**9.** $\log 112$.	**14.** $\log 147$.	**19.** $\log 7056$.
5. $\log 84$.	**10.** $\log 144$.	**15.** $\log 375$.	**20.** $\log 14406$.
6. $\log 45$.	**11.** $\log 216$.	**16.** $\log 343$.	**21.** $\log 15552$.

408. *In any system, the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ a^y = n \end{array}\right\}; \text{ whence, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_a n. \end{array}\right.$$

Dividing, we have $\frac{a^x}{a^y} = \frac{m}{n}$, or $a^{x-y} = \frac{m}{n}$.

Whence, $\log_a \frac{m}{n} = x - y.$

Substituting the values of x and y,

$$\log_a \frac{m}{n} = \log_a m - \log_a n.$$

409. 1. Given $\log 2 = .3010$; find $\log 5$.

$$\begin{aligned} \log 5 = \log \frac{10}{2} &= \log 10 - \log 2 \\ &= 1 - .3010 = .6990, \text{ } \textit{Ans.} \end{aligned}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$; find:

2. $\log \frac{7}{3}$.	5. $\log 35$.	8. $\log \frac{42}{25}$.	11. $\log 7\frac{1}{7}$.
3. $\log \frac{10}{7}$.	6. $\log \frac{21}{16}$.	9. $\log 175$.	12. $\log \frac{35}{6}$.
4. $\log 3\frac{1}{3}$.	7. $\log 125$.	10. $\log 11\frac{1}{9}$.	13. $\log 5\frac{4}{9}$.

410. *In any system, the logarithm of any power of a quantity is equal to the logarithm of the quantity multiplied by the exponent of the power.*

Assume the equation

$$a^x = m; \text{ whence, } x = \log_a m.$$

Raising both members to the pth power, we have

$$a^{px} = m^p; \text{ whence, } \log_a m^p = px = p \log_a m.$$

411. *In any system, the logarithm of any root of a quantity is equal to the logarithm of the quantity divided by the index of the root.*

For, $\log_a \sqrt[r]{m} = \log_a (m^{\frac{1}{r}}) = \frac{1}{r} \log_a m$ (Art. 410).

412. **1.** Given $\log 2 = .3010$; find the logarithm of $2^{\frac{5}{3}}$.

$$\log 2^{\frac{5}{3}} = \frac{5}{3} \times \log 2 = \frac{5}{3} \times .3010 = .5017, \textit{ Ans.}$$

Note. To multiply a logarithm by a fraction, multiply first by the numerator, and divide the result by the denominator.

2. Given $\log 3 = .4771$; find the logarithm of $\sqrt[8]{3}$.

$$\log \sqrt[8]{3} = \frac{\log 3}{8} = \frac{.4771}{8} = .0596, \textit{ Ans.}$$

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$; find:

3. $\log 3^{\frac{3}{5}}$.
4. $\log 2^9$.
5. $\log 7^5$.
6. $\log 5^{\frac{1}{5}}$.
7. $\log 12^{\frac{2}{3}}$.
8. $\log 21^{\frac{1}{2}}$.
9. $\log 14^4$.
10. $\log 25^{\frac{7}{3}}$.
11. $\log 15^{\frac{5}{6}}$.
12. $\log \sqrt{7}$.
13. $\log \sqrt[3]{3}$.
14. $\log \sqrt[7]{2}$.
15. $\log \sqrt[6]{5}$.
16. $\log \sqrt[4]{35}$.
17. $\log \sqrt[9]{98}$.
18. $\log \sqrt[12]{126}$.

19. Find the logarithm of $(2^{\frac{1}{3}} \times 3^{\frac{5}{4}})$.

By Art. 406, $\log (2^{\frac{1}{3}} \times 3^{\frac{5}{4}}) = \log 2^{\frac{1}{3}} + \log 3^{\frac{5}{4}}$

$$= \tfrac{1}{3} \log 2 + \tfrac{5}{4} \log 3$$

$$= .1003 + .5964 = .6967, \textit{ Ans.}$$

Find the values of the following:

20. $\log \left(\frac{10}{3}\right)^5$.
21. $\log \frac{7^{\frac{3}{4}}}{5^{\frac{2}{3}}}$.
22. $\log (3^{\frac{1}{6}} \times 2^{\frac{3}{5}})$.
23. $\log 3 \sqrt[4]{7}$.
24. $\log \sqrt{\frac{7}{3}}$.
25. $\log \frac{\sqrt[3]{7}}{\sqrt[5]{2}}$.
26. $\log \sqrt[3]{\frac{28}{5}}$.
27. $\log \frac{\sqrt{42}}{10^{\frac{2}{3}}}$.

413. *In the common system, the mantissæ of the logarithms of numbers having the same sequence of figures are equal.*

To illustrate, suppose that $\log 3.053 = .4847$; then,

$$\begin{aligned}
\log 30.53 &= \log(10 \times 3.053) = \log 10 + \log 3.053\\
&= 1 + .4847 \qquad\quad = 1.4847;\\
\log 305.3 &= \log(100 \times 3.053) = \log 100 + \log 3.053\\
&= 2 + .4847 \qquad\quad = 2.4847;\\
\log .03053 &= \log(.01 \times 3.053) = \log .01 + \log 3.053\\
&= 8 - 10 + .4847 \quad = 8.4847 - 10; \text{ etc.}
\end{aligned}$$

It is evident from the above that if a number is multiplied or divided by any integral power of 10, producing another number with the same sequence of figures, the mantissæ of their logarithms will be equal.

Thus, if $\log 3.053 = .4847$, then

$$\begin{aligned}
\log 30.53 &= 1.4847, & \log .3053 &= 9.4847 - 10,\\
\log 305.3 &= 2.4847, & \log .03053 &= 8.4847 - 10,\\
\log 3053. &= 3.4847, & \log .003053 &= 7.4847 - 10, \text{ etc.}
\end{aligned}$$

Note. The reason will now be seen for the statement made in Art. 402, that only the mantissæ are given in a table of logarithms of numbers. For, to find the logarithm of any number, we have only to take from the table the mantissa corresponding to its sequence of figures, and the characteristic may then be prefixed in accordance with the rules of Art. 402.

This property of logarithms is only enjoyed by the common system, and constitutes its superiority over others for the purposes of numerical computation.

414. 1. Given $\log 2 = .3010$, $\log 3 = .4771$; find $\log .00432$.

$$\begin{aligned}
\log 432 &= \log(2^4 \times 3^3) = 4\log 2 + 3\log 3\\
&= 1.2040 + 1.4313 = 2.6353.
\end{aligned}$$

Then by Art. 413, the *mantissa* of the result is .6353.

Whence by Art. 402, $\log .00432 = 7.6353 - 10$, *Ans.*

EXAMPLES.

Given $\log 2 = .3010$, $\log 3 = .4771$, and $\log 7 = .8451$; find:

2. $\log 1.8$.	**7.** $\log .0054$.	**12.** $\log 302.4$.
3. $\log 2.25$.	**8.** $\log .000315$.	**13.** $\log .06174$.
4. $\log .196$.	**9.** $\log 7350$.	**14.** $\log (8.1)^7$.
5. $\log .048$.	**10.** $\log 4.05$.	**15.** $\log \sqrt[5]{9.6}$.
6. $\log 38.4$.	**11.** $\log .448$.	**16.** $\log (22.4)^{\frac{1}{8}}$.

415. *To prove the relation*

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

Assume the equations

$$\left.\begin{array}{l} a^x = m \\ b^y = m \end{array}\right\}; \text{ whence, } \left\{\begin{array}{l} x = \log_a m, \\ y = \log_b m. \end{array}\right.$$

From the assumed equations, we have

$$a^x = b^y, \text{ or } a^{\frac{x}{y}} = b.$$

Whence, $\log_a b = \dfrac{x}{y}$, or $y = \dfrac{x}{\log_a b}$.

Substituting the values of x and y,

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

By aid of this relation, if the logarithm of a quantity m to a certain base a is known, its logarithm to any other base b may be found by dividing by the logarithm of b to the base a.

416. *To prove the relation*

$$\log_b a \times \log_a b = 1.$$

Putting $m = a$ in the result of Art. 415, we have

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b} \text{ (Art. 404)}.$$

Whence, $\log_b a \times \log_a b = 1.$

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

USE OF THE TABLE.

417. The table (pages 338 and 339) gives the mantissæ of the logarithms of all integers from 100 to 1000, calculated to four places of decimals.

418. *To find the logarithm of a number of three figures.*

Find in the column headed "No." the first two significant figures of the given number.

Then the mantissa required will be found in the corresponding horizontal line, in the vertical column headed by the third figure of the number.

Finally, prefix the characteristic by the rules of Art. 402.

For example, $\log 168 = 2.2253$;

$$\log .344 = 9.5366 - 10; \text{ etc.}$$

419. For a number consisting of one or two significant figures, the column headed 0 may be used.

Thus, let it be required to find $\log 83$ and $\log 9$.

By Art. 413, $\log 83$ has the same mantissa as $\log 830$, and $\log 9$ the same mantissa as $\log 900$. Hence,

$$\log 83 = 1.9191, \text{ and } \log 9 = 0.9542.$$

420. *To find the logarithm of a number of more than three figures.*

1. Required the logarithm of 327.6.

We find from the table, $\log 327 = 2.5145$,

$$\log 328 = 2.5159.$$

That is, an increase of one unit in the number produces an increase of .0014 in the logarithm.

Therefore an increase of .6 of a unit in the number will produce an increase of $.6 \times .0014$ in the logarithm, or .0008 to the nearest fourth decimal place.

Hence, $\log 327.6 = 2.5145 + .0008 = 2.5153.$

Note. The difference between any mantissa in the table and the mantissa of the next higher number of three figures, is called the *tabular difference.* The subtraction may be performed mentally.

The following rule is derived from the above:

Find from the table the mantissa of the first three significant figures, and the tabular difference.

Multiply the latter by the remaining figures of the number, with a decimal point before them.

Add the result to the mantissa of the first three figures, and prefix the proper characteristic.

EXAMPLES.

2. Find the logarithm of .021508.

$$\begin{array}{lr} & \text{Mantissa of } 215 = 3324 \\ \text{Tabular difference} = \ \ 21 & 2 \\ \underline{\quad .08} & \overline{3326} \\ \text{Correction} = 1.68 = 2, \text{ nearly.} & \end{array}$$

Ans. 8.3326 − 10.

Find the logarithms of the following:

3. 80.	**7.** .7723.	**11.** 20.08.	**15.** 5.1809.
4. 6.3.	**8.** 1056.	**12.** 92461.	**16.** 1036.5.
5. 298.	**9.** 3.294.	**13.** .40322.	**17.** .086676.
6. .902.	**10.** .05205.	**14.** .007178.	**18.** .11507.

421. *To find the number corresponding to a logarithm.*

1. Required the number whose logarithm is 1.6571.

Find in the table the mantissa 6571.

In the corresponding line, in the column headed "No.,' we find 45, the first two figures of the required number, and at the head of the column we find 4, the third figure.

Since the characteristic is 1, there must be two figures to the left of the decimal point (Art. 402).

Hence, number corresponding to 1.6571 = 45.4, *Ans.*

2. Required the number whose logarithm is 2.3934.

We find in the table the mantissæ 3927 and 3945, whose corresponding numbers are 247 and 248, respectively.

That is, an increase of 18 in the mantissa produces an increase of one unit in the number corresponding.

Therefore, an increase of 7 in the mantissa will produce an increase of $\frac{7}{18}$ of a unit in the number, or .39, nearly.

Hence, number corresponding $= 247 + .39 = 247.39$, *Ans.*

The following rule is derived from the above:

Find from the table the next less mantissa, the three figures corresponding, and the tabular difference.

Subtract the next less from the given mantissa, and divide the remainder by the tabular difference.

Annex the quotient to the first three figures of the number, and point off the result.

Note. The rules for pointing off are the reverse of those of Art. 402:

I. *If* -10 *is not written after the mantissa, add* 1 *to the characteristic, giving the number of places to the left of the decimal point.*

II. *If* -10 *is written after the mantissa, subtract the positive part of the characteristic from* 9, *giving the number of ciphers between the decimal point and first significant figure.*

EXAMPLES.

3. Find the number whose logarithm is $8.5264 - 10$.

$$\begin{array}{l} \qquad\qquad\qquad\quad 5264 \\ \text{Next less mantissa} = \underline{5263}\,;\ \text{three figures corresponding} = 336. \\ \text{Tabular difference} = 13)1.000(.077 = .08,\ \text{nearly.} \\ \qquad\qquad\qquad\qquad\quad \underline{91} \\ \qquad\qquad\qquad\qquad\quad\ 90 \end{array}$$

According to the above rule, there will be one cipher between the decimal point and first significant figure.

Hence, number corresponding $= .033608$, *Ans.*

Find the numbers corresponding to the following logarithms:

4. 1.8055.	**9.** 8.1648 − 10.	**14.** 1.6482.
5. 9.4487 − 10.	**10.** 7.5209 − 10.	**15.** 7.0450 − 10.
6. 0.2165.	**11.** 4.0095.	**16.** 4.8016.
7. 3.9487.	**12.** 0.9774.	**17.** 8.1144 − 10.
8. 2.7364.	**13.** 9.3178 − 10.	**18.** 2.7015.

APPLICATIONS.

422. The value of an arithmetical quantity, in which the operations indicated involve only multiplication, division, involution, or evolution, may be most conveniently found by logarithms.

The utility of the process consists in the fact that addition takes the place of multiplication, subtraction of division, multiplication of involution, and division of evolution.

Note. In computations with four-place logarithms, the results cannot usually be depended upon to more than *four* significant figures.

423. 1. Find the value of $.0631 \times 7.208 \times .51272$.

By Art. 406, $\log(.0631 \times 7.208 \times .51272)$
$$= \log .0631 + \log 7.208 + \log .51272.$$

$$\begin{aligned} \log\ .0631 &= 8.8000 - 10 \\ \log\ 7.208 &= 0.8578 \\ \log .51272 &= \underline{9.7099 - 10} \end{aligned}$$

Adding, $\therefore$ log of result $= 19.3677 - 20$
$$= 9.3677 - 10 \text{ (see Note 1)}$$

Number corresponding to $9.3677 - 10 = .2332$, *Ans.*

Note 1. If the sum is a negative logarithm, it should be reduced so that the negative portion of the characteristic may be −10.

Thus, 19.3677 − 20 is reduced to 9.3677 − 10.

2. Find the value of $\frac{336.8}{7984}$.

By Art. 408, $\log\frac{336.8}{7984} = \log 336.8 - \log 7984.$

$$\begin{aligned} \log 336.8 &= 12.5273 - 10 \quad \text{(see Note 2)} \\ \log\ 7984 &= \underline{\ \ 3.9022\qquad\quad} \end{aligned}$$

Subtracting, $\therefore$ log of result $= \ \ 8.6251 - 10$

Number corresponding $= .04218$, *Ans.*

Note 2. To subtract a greater logarithm from a less, or to subtract a negative logarithm from a positive, increase the characteristic of the minuend by 10, writing -10 after the mantissa to compensate.

Thus, to subtract 3.9022 from 2.5273, write the minuend in the form $12.5273 - 10$; subtracting 3.9022 from this, the result is $8.6251 - 10$.

3. Find the value of $(.07396)^5$.

By Art. 410, $\log(.07396)^5 = 5 \times \log .07396.$

$$\begin{aligned} \log .07396 &= 8.8690 - 10 \\ &\qquad\qquad\quad\ \underline{\ \ 5} \\ &\ \ \ 44.3450 - 50 \\ &= 4.3450 - 10 \quad \text{(see Note 1)} \\ &= \log .000002213, \ \textit{Ans.} \end{aligned}$$

4. Find the value of $\sqrt[3]{.035063}$.

By Art. 411, $\log \sqrt[3]{.035063} = \frac{1}{3}\log .035063.$

$$\begin{aligned} \log .035063 &= 8.5449 - 10 \\ &\quad\ \ 20. \qquad\ -20 \quad \text{(see Note 3)} \\ &3\overline{)28.5449 - 30} \\ &\quad\ \ 9.5150 - 10 \\ &= \log .3274, \ \textit{Ans.} \end{aligned}$$

Note 3. To divide a negative logarithm, add to both parts such a multiple of 10 as will make the negative portion of the characteristic exactly divisible by the divisor, with -10 as the quotient.

Thus, to divide $8.5449 - 10$ by 3, add 20 to both parts of the logarithm, giving the result $28.5449 - 30$. Dividing this by 3, the quotient is $9.5150 - 10$.

ARITHMETICAL COMPLEMENT.

424. The *Arithmetical Complement* of the logarithm of a number, or, briefly, the *Cologarithm* of the number, is the logarithm of the reciprocal of that number.

Thus, $\text{colog}\,409 = \log\frac{1}{409} = \log 1 - \log 409.$

$$\begin{aligned} \log 1 &= 10. \qquad -10 \quad \text{(Note 2, Art. 423)} \\ \log 409 &= \underline{\ 2.6117\qquad\ } \\ \therefore \text{colog}\,409 &= \ 7.3883 - 10. \end{aligned}$$

Again, $\text{colog}\,.067 = \log\frac{1}{.067} = \log 1 - \log .067.$

$$\begin{aligned} \log 1 &= 10. \qquad -10 \\ \log .067 &= \underline{\ 8.8261 - 10} \\ \therefore \text{colog}\,.067 &= \ 1.1739 \end{aligned}$$

The following rule is evident from the above:

To find the cologarithm of a number, subtract its logarithm from $10 - 10$.

Note. The cologarithm may be obtained from the logarithm by subtracting the last *significant* figure from 10 and each of the others from 9, -10 being written after the result in the case of a positive logarithm.

425. *Example.* Find the value of $\frac{.51384}{8.709 \times .0946}$.

$$\begin{aligned} \log\frac{.51384}{8.709 \times .0946} &= \log\left(.51384 \times \frac{1}{8.709} \times \frac{1}{.0946}\right) \\ &= \log .51384 + \log\frac{1}{8.709} + \log\frac{1}{.0946} \\ &= \log .51384 + \text{colog}\,8.709 + \text{colog}\,.0946. \end{aligned}$$

$$\begin{aligned} \log .51384 &= 9.7109 - 10 \\ \text{colog}\,8.709 &= 9.0601 - 10 \\ \text{colog}\,.0946 &= \underline{1.0241\qquad\quad} \\ & \ 9.7951 - 10 = \log .6239, \textit{ Ans.} \end{aligned}$$

It is evident from the above that the logarithm of a fraction is equal to the logarithm of the numerator *plus* the cologarithm of the denominator.

Or in general, to find the logarithm of a fraction whose terms are composed of factors,

Add together the logarithms of the factors of the numerator, and the cologarithms of the factors of the denominator.

Note. The value of the above fraction may be found without using cologarithms, by the following formula:

$$\log \frac{.51384}{8.709 \times .0946} = \log .51384 - \log (8.709 \times .0946)$$
$$= \log .51384 - (\log 8.709 + \log .0946).$$

The advantage in the use of cologarithms is that the written work of computation is exhibited in a more compact form.

EXAMPLES.

426. Note. A *negative* quantity can have no common logarithm, as is evident from the definition of Art. 395. If negative quantities occur in computation, they may be treated as if they were positive, and the *sign* of the result determined irrespective of the logarithmic work.

Thus, in Ex. 3, p. 346, the value of $721.3 \times (-3.0528)$ may be obtained by finding the value of 721.3×3.0528, and putting a negative sign before the result. See also Ex. 34, p. 347.

Find by logarithms the values of the following:

1. $9.238 \times .9152$.
2. $130.36 \times .08237$.
3. $721.3 \times (-3.0528)$.
4. $(-4.3264) \times (-.050377)$.
5. $.27031 \times .042809$.
6. $(-.063165) \times 11.134$.
7. $\dfrac{401.8}{52.37}$.
8. $\dfrac{7.2321}{10.813}$.
9. $\dfrac{-.3384}{.08659}$.
10. $\dfrac{9.163}{.0051422}$.
11. $\dfrac{22518}{64327}$.
12. $\dfrac{.007514}{-.015822}$.
13. $\dfrac{3.3681}{12.853 \times .6349}$.
14. $\dfrac{15.008 \times (-.0843)}{.06376 \times 4.248}$.

15. $\dfrac{(-2563) \times .03442}{714.8 \times (-.511)}$.

16. $\dfrac{121.6 \times (-9.025)}{(-48.3) \times 3662 \times (-.0856)}$.

17. $(23.86)^3$.

18. $(.532)^8$.

19. $(-1.0246)^7$.

20. $(.09323)^5$.

21. $5^{\frac{2}{3}}$.

22. $(.8)^{\frac{2}{5}}$.

23. $(-3.16)^{\frac{4}{5}}$.

24. $(.021)^{\frac{5}{2}}$.

25. $\sqrt{2}$.

26. $\sqrt[4]{5}$.

27. $\sqrt[5]{-3}$.

28. $\sqrt{.4294}$.

29. $\sqrt[3]{.02305}$.

30. $\sqrt[8]{1000}$.

31. $\sqrt[7]{-.00951}$.

32. $\sqrt[5]{.0001011}$.

33. Find the value of $\dfrac{2\sqrt[3]{5}}{3^{\frac{5}{6}}}$.

$$\log \frac{2\sqrt[3]{5}}{3^{\frac{5}{6}}} = \log 2 + \log \sqrt[3]{5} + \operatorname{colog} 3^{\frac{5}{6}} \text{ (Art. 425)}$$
$$= \log 2 + \tfrac{1}{3}\log 5 + \tfrac{5}{6}\operatorname{colog} 3.$$

$$\log 2 = \quad .3010$$

$\log 5 = .6990$; divide by $3 = .2330$

$\operatorname{colog} 3 = 9.5229 - 10$; multiply by $\frac{5}{6} = 9.6024 - 10$

$$.1364$$
$$= \log 1.369, \textit{ Ans.}$$

34. Find the value of $\sqrt[3]{\dfrac{-.03296}{7.962}}$.

$$\log \sqrt[3]{\frac{.03296}{7.962}} = \tfrac{1}{3}\log\frac{.03296}{7.962} = \tfrac{1}{3}(\log .03296 - \log 7.962).$$

$\log .03296 = 8.5180 - 10$

$\log 7.962 = 0.9010$

$3\overline{)27.6170 - 30}$

$9.2057 - 10 = \log .1606.$

Ans. $-.1606$.

Find the values of the following:

35. $2^{\frac{3}{2}} \times 3^{\frac{2}{3}}$.

36. $\dfrac{3^{\frac{5}{6}}}{4^{\frac{2}{3}}}$.

37. $\dfrac{5^{\frac{3}{7}}}{(-10)^{\frac{2}{9}}}$.

38. $\left(\dfrac{6}{7}\right)^{\frac{5}{2}}$.

39. $\left(\dfrac{35}{113}\right)^{\frac{2}{3}}$.

40. $\left(\dfrac{.08726}{.1321}\right)^{\frac{5}{3}}$.

41. $\sqrt[8]{\dfrac{21}{13}}$.

42. $\sqrt[9]{-\dfrac{3}{7}}$.

43. $\sqrt[5]{\dfrac{2}{3}} \div \sqrt[3]{\dfrac{3}{5}}$.

44. $\sqrt[8]{2} \times \sqrt[5]{3} \times \sqrt[7]{.01}$.

45. $\sqrt[5]{\dfrac{3258}{49309}}$.

46. $\left(\dfrac{-31.63}{429}\right)^{\frac{3}{17}}$.

47. $\dfrac{100^{\frac{2}{3}}}{(.7325)^{\frac{3}{7}}}$.

48. $\dfrac{\sqrt[3]{.0001289}}{\sqrt[4]{.0008276}}$.

49. $\dfrac{(-.7469)^{\frac{5}{3}}}{-(.2345)^{\frac{7}{2}}}$.

50. $\dfrac{\sqrt[11]{.0073}}{(.68291)^{\frac{5}{4}}}$.

51. $\dfrac{\sqrt{5.955} \times \sqrt[3]{61.2}}{\sqrt[5]{298.54}}$.

52. $(538.2 \times .0005969)^{\frac{1}{5}}$.

53. $(18.9503)^{11} \times (-.1)^{14}$.

54. $\sqrt[6]{3734.9 \times .00001108}$.

55. $(2.6317)^{\frac{3}{4}} \times (.71272)^{\frac{2}{5}}$.

56. $\dfrac{\sqrt[3]{-.008193} \times (.06285)^{\frac{3}{2}}}{-.98342}$.

57. $\sqrt{.035} \times \sqrt[6]{.62667} \times \sqrt[3]{.0072103}$.

EXPONENTIAL EQUATIONS.

427. An **Exponential Equation** is one in which the unknown quantity occurs as an exponent.

To solve an equation of this form, take the logarithms of both members; the result will be an equation which can be solved by ordinary algebraic methods.

1. Given $31^x = 23$; find the value of x.

Taking the logarithms of both members,

$$\log(31^x) = \log 23.$$

Or, by Art. 410,

$$x \log 31 = \log 23.$$

Whence, $$x = \frac{\log 23}{\log 31} = \frac{1.3617}{1.4914} = .91303, \textit{ Ans.}$$

2. Given $.2^x = 3$; find the value of x.

Taking the logarithms of both members,

$$x \log .2 = \log 3.$$

Whence, $$x = \frac{\log 3}{\log .2} = \frac{.4771}{9.3010 - 10} = \frac{.4771}{-.6990}$$
$$= -.6825, \textit{ Ans.}$$

EXAMPLES.

Solve the following equations:

3. $11^x = 3$. **5.** $13^x = .281$. **7.** $a^x = b^m c^n$.

4. $.3^x = .8$. **6.** $.703^x = 1.096$. **8.** $ma^{\frac{1}{x}} = n$.

9. $21^{x^2-2x} = 9260$. **10.** $.051^{3x+7} = 384.4$.

11. Given a, r, and l; derive the formula for n. (Art. 350.)

12. Given a, r, and S; derive the formula for n.

13. Given a, l, and S; derive the formula for n.

14. Given r, l, and S; derive the formula for n.

15. Find the logarithm of .3 to the base 7.

By Art. 415, we have

$$\log_7 .3 = \frac{\log_{10} .3}{\log_{10} 7} = \frac{9.4771 - 10}{.8451} = -\frac{.5229}{.8451} = -.6187, \textit{ Ans.}$$

Find the values of the following:

16. $\log_2 11$. **18.** $\log_{.3} 65$. **20.** $\log_{73} 56.31$.

17. $\log_5 .8$. **19.** $\log_{.8} .0823$. **21.** $\log_{15} .007228$.

EXPONENTIAL AND LOGARITHMIC SERIES.

428. We have for all values of n and x,

$$\left[\left(1+\frac{1}{n}\right)^n\right]^x=\left(1+\frac{1}{n}\right)^{nx}.$$

Expanding both members by the Binomial Theorem,

$$\left[1+n\frac{1}{n}+\frac{n(n-1)}{\lfloor 2}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{\lfloor 3}\frac{1}{n^3}+\cdots\right]^x$$

$$=1+nx\frac{1}{n}+\frac{nx(nx-1)}{\lfloor 2}\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\lfloor 3}\frac{1}{n^3}+\cdots$$

That is,

$$\left[1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\cdots\right]^x$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\cdots \qquad (1)$$

This equation holds however great n may be.

Now let n be indefinitely increased.

Then since each of the terms $\frac{1}{n}$, $\frac{2}{n}$, $\cdots$, approaches the limit 0 (Art. 301), the limit of the first member of (1) is

$$\left[1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\right]^x,$$

and the limit of the second member is

$$1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots$$

By the Theorem of Limits (Art. 299) these limits are equal; that is,

$$\left[1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\right]^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots$$

Denoting the series in brackets by e, we obtain

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots \qquad (2)$$

429. Substituting mx in place of x in (2), Art. 428, we have

$$e^{mx} = 1 + mx + \frac{m^2x^2}{\underline{|2}} + \frac{m^3x^3}{\underline{|3}} + \cdots \quad (3)$$

Let $e^m = a$; then $m = \log_e a$ (Art. 395), and $e^{mx} = a^x$.

Substituting in (3), we obtain

$$a^x = 1 + (\log_e a)\,x + (\log_e a)^2\frac{x^2}{\underline{|2}} + (\log_e a)^3\frac{x^3}{\underline{|3}} + \cdots \quad (4)$$

This result is called the *Exponential Theorem.*

430. The system of logarithms which has e for its base is called the *Napierian System*, from Napier, the inventor of logarithms.

The approximate value of e may be readily calculated by aid of the series of Art. 428,

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \cdots$$

and will be found to equal 2.7182818...

431. *To expand* $\log_e(1+x)$ *in ascending powers of* x.

Substituting in equation (4), Art. 429, $1+x$ in place of a, and y in place of x, we obtain

$$(1+x)^y = 1 + [\log_e(1+x)]y + \text{terms in } y^2,\ y^3, \text{ etc.}$$

Expanding the first member by the Binomial Theorem,

$$1 + yx + \frac{y(y-1)}{\underline{|2}}x^2 + \frac{y(y-1)(y-2)}{\underline{|3}}x^3 + \cdots$$
$$= 1 + [\log_e(1+x)]\,y + \text{terms in } y^2,\ y^3, \text{ etc.}$$

This equation holds for every value of y which makes both members convergent, and by the Theorem of Undetermined Coefficients the coefficients of y in the two series are equal.

That is, $$x - \frac{\underline{|1}}{\underline{|2}}x^2 + \frac{\underline{|2}}{\underline{|3}}x^3 - \frac{\underline{|3}}{\underline{|4}}x^4 + \cdots = \log_e(1+x);$$

or, $$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

432. The above formula can be used for the calculation of Napierian logarithms if x is so taken that the series in the second member is convergent; but unless x is small, it requires the sum of a great many terms to ensure any degree of accuracy.

433. *To derive a more convenient formula for calculating the Napierian logarithm of a number.*

By Art. 431, we have

$$\log_e(1+x) = \quad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \quad (1)$$

Putting $-x$ in place of x, this becomes

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots \quad (2)$$

Subtracting (2) from (1), we have

$$\log_e(1+x) - \log_e(1-x) = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots$$

Whence, by Art. 408,

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right). \quad (3)$$

Let $\quad x = \dfrac{m-n}{m+n}$; then $\dfrac{1+x}{1-x} = \dfrac{1+\dfrac{m-n}{m+n}}{1-\dfrac{m-n}{m+n}} = \dfrac{2m}{2n} = \dfrac{m}{n}.$

Substituting these values in (3), we obtain

$$\log_e \frac{m}{n} = 2\left[\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \cdots\right].$$

But by Art. 408, $\log_e \dfrac{m}{n} = \log_e m - \log_e n.$

Whence,

$$\log_e m = \log_e n + 2\left[\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \cdots\right].$$

434. Let it be required, for example, to calculate the Napierian logarithm of 2 to six places of decimals.

Putting $m = 2$ and $n = 1$ in the result of Art. 433, we have

$$\log_e 2 = \log_e 1 + 2\left[\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \cdots\right].$$

Or, since $\log_e 1 = 0$ (Art. 403),

$$\log_e 2 = 2(.3333333 + .0123457 + .0008230 + .0000653 + .0000056 + .0000005 + \cdots)$$

$$= 2 \times .3465734 = .6931468$$

$$= .693147,\text{ correct to the sixth place of decimals.}$$

Having found $\log_e 2$, we may calculate $\log_e 3$ by putting $m = 3$ and $n = 2$ in the result of Art. 433.

Proceeding in this way, we shall find $\log_e 10 = 2.302585\ldots$

435. *To calculate the common logarithm of a number, having given its Napierian logarithm.*

Putting $b = 10$ and $a = e$ in the result of Art. 415, we have

$$\log_{10} m = \frac{\log_e m}{\log_e 10} = \frac{1}{2.302585} \times \log_e m = .4342945 \times \log_e m.$$

Thus, $\log_{10} 2 = .4342945 \times .693147 = .301030.$

436. The multiplier by which logarithms of any system are derived from Napierian logarithms, is called the *modulus* of that system.

Thus, .4342945 is the modulus of the common system.

437. *Conversely,* to find the Napierian logarithm of a number when its common logarithm is given, we may either divide the common logarithm by the modulus .4342945, or multiply it by 2.302585, the reciprocal of .4342945.

EXAMPLES.

Find the Napierian logarithms of the following:

1. 100. **3.** 88.2. **5.** .343.

2. .0001. **4.** 1325. **6.** .08562.

XXXVII. COMPOUND INTEREST AND ANNUITIES.

438. The principles of logarithms may be applied to the solution of problems in Compound Interest.

Let $P =$ the principal in dollars;

$n =$ the number of years;

$t =$ the ratio to one year of the time during which simple interest is calculated; for instance, if the interest is compounded semi-annually, $t = \frac{1}{2}$;

$R =$ the amount of one dollar for the time t;

$A =$ the amount of P dollars for n years.

1. *Given P, n, t, R; to find A.*

Since the amount of one dollar for the time t is R, the amount of P dollars for the same period will be PR.

That is, the amount at the end of the 1st interval is PR.

In like manner, the amount at the end of the

$$\text{2nd interval} = PR \times R = PR^2;$$

$$\text{3rd interval} = PR^2 \times R = PR^3; \text{ etc.}$$

Since the whole number of intervals is $\frac{n}{t}$, the amount at the end of the last one, in accordance with the law observed above, will be $PR^{\frac{n}{t}}$.

That is, $$A = PR^{\frac{n}{t}}. \qquad (1)$$

By logarithms, $$\log A = \log P + \frac{n}{t}\log R. \qquad (2)$$

Example. What will be the amount of $7326 for 3 years and 9 months at 7 per cent compound interest, the interest being compounded quarterly?

In this case,

$$P = 7326,\ n = 3\tfrac{3}{4},\ t = \tfrac{1}{4},\ R = 1.0175, \text{ and } \frac{n}{t} = 15.$$

$$\begin{aligned} \log P &= 3.8649 \\ \log R = 0.0075;\ \text{multiply by } 15 &= \underline{0.1125} \\ \log A &= 3.9774 \\ \therefore A &= \$9492, \textit{ Ans.} \end{aligned}$$

2. *Given* n, t, R, A; *to find* P.

From (2), $$\log P = \log A - \frac{n}{t} \log R.$$

Example. What sum of money will amount to \$1763.50 in 3 years at 5 per cent compound interest, the interest being compounded semi-annually?

In this case,

$$n = 3,\ t = \tfrac{1}{2},\ R = 1.025,\ A = 1763.5, \text{ and } \frac{n}{t} = 6.$$

$$\begin{aligned} \log A &= 3.2464 \\ \log R = 0.0107;\ \text{multiply by } 6 &= \underline{0.0642} \\ \log P &= 3.1822 \\ \therefore P &= \$1521.40, \textit{ Ans.} \end{aligned}$$

3. *Given* P, t, R, A; *to find* n.

From (2), $\dfrac{n}{t} \log R = \log A - \log P.$

Whence, $$n = \frac{t(\log A - \log P)}{\log R}.$$

Example. In how many years will \$300 amount to \$396.90 at 6 per cent compound interest, the interest being compounded quarterly?

Here, $P = 300$, $t = \frac{1}{4}$, $R = 1.015$, and $A = 396.9$.

$$\therefore n = \frac{\log 396.9 - \log 300}{4 \log 1.015} = \frac{2.5987 - 2.4771}{4 \times .0064} = \frac{.1216}{.0256}$$

$$= 4.75 \text{ years, } \textit{Ans.}$$

4. *Given P, n, t, A; to find R.*

From (2), $$\log R = \frac{\log A - \log P}{\frac{n}{t}}.$$

Example. At what rate per cent per annum will \$500 amount to \$688.83 in 6 years and 6 months, the interest being compounded semi-annually?

Here, $P = 500$, $n = 6\frac{1}{2}$, $t = \frac{1}{2}$, $A = 688.83$, and $\frac{n}{t} = 13$.

$$\begin{array}{rr} \log A = & 2.8381 \\ \log P = & 2.6990 \\ & 13\overline{)0.1391} \\ \log R = & 0.0107 \\ \therefore R = & 1.025. \end{array}$$

That is, the *interest* on one dollar for 6 months is \$.025, and the rate is 5 per cent per annum.

EXAMPLES.

439. 1. What will be the amount of \$1000 for 18 years at 6 per cent compound interest, the interest being compounded annually?

2. What sum of money will amount to \$870.50 in 7 years and 3 months at 3 per cent compound interest, the interest being compounded quarterly?

3. In how many years will \$968 amount to \$1269.40 at 5 per cent compound interest, the interest being compounded semi-annually?

4. At what rate per cent per annum will \$2600 gain \$416.40 in 3 years and 9 months, the interest being compounded quarterly?

5. In how many years will a sum of money double itself at 5 per cent compound interest, the interest being compounded annually?

6. In how many years will a sum of money treble itself at 7 per cent compound interest, the interest being compounded semi-annually?

7. What sum of money will amount to \$1000 in 11 years and 8 months at $3\frac{3}{4}$ per cent compound interest, the interest being compounded every four months?

ANNUITIES.

440. The *present value* of a sum of money, due at the end of a given period, is the sum which when put at interest for the period in question will amount to the given sum.

In finding the present value of an annuity, it is customary to allow compound interest.

441. *To find the present value of an annuity to continue for n successive years, allowing compound interest.*

Let $A =$ the annuity in dollars;
$R =$ the amount of one dollar for one year;
$P_m =$ the present value of the payment due at the end of m years;
$P =$ the present value of the annuity.

By Art. 440, the sum P_m will amount to A when put at compound interest for m years, the interest being compounded annually.

In this case, $n = m$, and $t = 1$; whence by (1), Art. 438,

$$A = P_m R^m, \text{ or } P_m = \frac{A}{R^m}.$$

By aid of the above formula, the present value of the

$$\text{1st payment} = \frac{A}{R};$$

$$\text{2nd payment} = \frac{A}{R^2};$$

. . . .

$$n\text{th payment} = \frac{A}{R^n}.$$

Hence the sum of the present values of the separate payments, or the present value of the annuity, is

$$\frac{A}{R^n}+\frac{A}{R^{n-1}}+\cdots+\frac{A}{R^2}+\frac{A}{R}.$$

That is, $$P=A\left[\frac{1}{R^n}+\frac{1}{R^{n-1}}+\cdots+\frac{1}{R^2}+\frac{1}{R}\right].$$

The expression in brackets is the sum of the terms of a Geometrical Progression, in which $a=\frac{1}{R^n}$, $r=R$, and $l=\frac{1}{R}$; whence by (II.), Art. 348,

$$P=A\frac{1-\frac{1}{R^n}}{R-1}. \qquad (1)$$

Example. What is the present value of an annuity of \$150 to continue for 20 years, allowing 4 per cent compound interest?

Here, $A=150$, $n=20$, $R=1.04$, and $R-1=.04$.

Whence, $$P=\frac{150}{.04}\left[1-\frac{1}{(1.04)^{20}}\right].$$

$$\log\frac{1}{(1.04)^{20}}=20 \text{ colog } 1.04.$$

$$\begin{aligned}\text{colog } 1.04 &= 9.9830\\ &\underline{\quad\quad 20}\\ &9.6600\end{aligned}$$

Number corresponding $=$ .4571.

Therefore, $$P=\frac{150}{.04}(1-.4571)=3750\times .5429.$$

$$\begin{aligned}\log 3750 &= 3.5740\\ \log .5429 &= \underline{9.7347}\\ \log P &= 3.3087\\ \therefore P &= \$2035.70, \textit{ Ans.}\end{aligned}$$

442. If in (1), Art. 441, n is indefinitely increased, the limiting value of the second member is

$$\frac{A}{R-1} \text{ (Art. 301)}.$$

That is, *the present value of a perpetual annuity is equal to the amount of the annuity divided by the interest on one dollar for one year.*

EXAMPLES.

443. 1. What is the present value of an annuity of \$200 to continue 15 years, allowing 5 per cent compound interest?

2. What is the present value of a perpetual annuity of \$600, allowing $3\frac{1}{2}$ per cent compound interest?

3. What is the present value of an annuity of \$1127 to continue 3 years, allowing 7 per cent compound interest?

4. What annuity to continue 10 years can be purchased for \$2038, allowing 6 per cent compound interest?

5. A person borrows \$5254; how much must he pay in annual instalments in order that the whole debt may be discharged in 12 years, allowing $4\frac{1}{2}$ per cent compound interest?

XXXVIII. PERMUTATIONS AND COMBINATIONS.

444. The different orders in which quantities can be arranged are called their **Permutations.**

Thus the permutations of the quantities a, b, c, taken *two* at a time, are

$$ab,\ ac,\ ba,\ bc,\ ca,\ cb;$$

and their permutations taken *three* at a time, are

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba.$$

445. The **Combinations** of quantities are the different collections which can be formed with them, without regard to the order in which they are placed.

Thus the combinations of the quantities a, b, c, taken *two* at a time, are

$$ab,\ bc,\ ca;$$

for though ab and ba are different permutations, they form the same combination.

446. *To find the number of permutations of n quantities taken two at a time.*

Let the quantities be $a_1, a_2, a_3, a_4, \ldots, a_n$.

The permutations of the quantities taken two at a time, having a_1 as the first element, are

$$a_1a_2,\ a_1a_3,\ a_1a_4, \ldots,\ a_1a_n;$$

the number of which is $n-1$.

In like manner, there are $n-1$ permutations of the quantities taken two at a time, having a_2 as the first element; and similarly for each of the remaining quantities $a_3, a_4, \ldots, a_n$.

Therefore the whole number of permutations of the quantities taken two at a time is equal to

$$n(n-1).$$

447. We will now consider the general case.

To find the number of permutations of n quantities taken r at a time.

Let the quantities be

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_r,\ a_{r+1},\ a_{r+2},\ \ldots,\ a_n.$$

One of the permutations containing r quantities will be that consisting of the first r quantities in their order; that is,

$$a_1 a_2 a_3 \ldots a_r.$$

Placing after this the other $n-r$ quantities one at a time, as follows,

$$a_1 a_2 a_3 \ldots a_r a_{r+1}$$
$$a_1 a_2 a_3 \ldots a_r a_{r+2}$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$a_1 a_2 a_3 \ldots a_r a_n$$

there are formed $n-r$ different permutations, each containing $r+1$ quantities.

We may proceed in a similar manner with the remaining permutations containing r quantities, and in each case we shall obtain $n-r$ permutations containing $r+1$ quantities.

That is, the number of permutations of the quantities taken r at a time, multiplied by $n-r$, is equal to the number of permutations of the quantities taken $r+1$ at a time.

But the number of permutations of the quantities taken *two* at a time is equal to $n(n-1)$ (Art. 446).

Hence the number of permutations of the quantities taken *three* at a time, is equal to the number taken *two* at a time, multiplied by $n-2$, or $n(n-1)(n-2)$.

The number of permutations of the quantities taken *four* at a time, is equal to the number taken *three* at a time, multiplied by $n-3$, or $n(n-1)(n-2)(n-3)$; and so on.

We observe that the last factor in the number of permutations is n, minus a number one less than the number of quantities taken at a time.

Hence the number of permutations of the quantities taken r at a time is given by the formula

$$n(n-1)(n-2)\cdots[n-(r-1)],$$

or, $$n(n-1)(n-2)\cdots(n-r+1). \qquad (1)$$

448. If *all* the quantities are taken together, $r=n$, and formula (1) becomes

$$n(n-1)(n-2)\cdots 1 = \underline{|n}. \qquad (2)$$

That is, *the number of permutations of n quantities taken n at a time is equal to the product of the natural numbers from 1 to n inclusive.* (See Note 2, Art. 363.)

449. *To find the number of combinations of n quantities taken r at a time.*

The number of *permutations* of n quantities taken r at a time is

$$n(n-1)(n-2)\cdots(n-r+1) \text{ (Art. 447)}.$$

But by Art. 448, each combination of r quantities may have $\underline{|r}$ permutations.

Hence the number of *combinations* of n quantities taken r at a time is equal to the number of permutations, divided by $\underline{|r}$; that is,

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}}. \qquad (3)$$

450. Multiplying both terms of (3) by the product of the natural numbers from 1 to $n-r$ inclusive, we have

$$\frac{n(n-1)\cdots(n-r+1)\times(n-r)\cdots 3\cdot 2\cdot 1}{\underline{|r}\times 1\cdot 2\cdot 3\cdots(n-r)} = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}};$$

which is another form of the result.

451. By Art. 450, the number of combinations of n quantities taken $n-r$ at a time, is

$$\frac{\underline{|n}}{\underline{|n-r}\,\underline{|n-(n-r)}}, \text{ or } \frac{\underline{|n}}{\underline{|n-r}\,\underline{|r}}.$$

But this is the same as the number of combinations of n quantities taken r at a time (Art. 450).

Hence, *the number of combinations of n quantities taken r at a time is equal to the number of combinations of n quantities taken $n-r$ at a time.*

EXAMPLES.

452. 1. How many changes can be rung with ten bells, taking 7 at a time?

Here $n=10$, $r=7$, and $n-r+1=4$.

Then by (1), Art. 447, the required number

$$=10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4=604800, \textit{ Ans.}$$

2. How many different combinations can be formed with 16 letters, taking 12 at a time?

By Art. 451, the number of combinations of 16 quantities taken 12 at a time is equal to the number of combinations of 16 quantities taken 4 at a time.

Putting $n=16$ and $r=4$, in (3), Art. 449, we have

$$\frac{16\cdot 15\cdot 14\cdot 13}{1\cdot 2\cdot 3\cdot 4}=1820, \textit{ Ans.}$$

3. How many permutations can be formed of the 26 letters of the alphabet, taken 5 at a time?

4. How many permutations can be formed of the letters in the word *forming*, taken all together?

5. How many combinations can be formed with the letters in the word *triangles*, taking four at a time?

6. How many different numbers, of five different figures each, can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9?

7. From a company of 40 soldiers, how many different pickets of 6 men can be taken?

8. How many combinations can be formed with 18 quantities, taking 11 at a time?

9. How many different words of 4 letters each can be made with 6 letters? How many of 3 letters each? How many of 6 letters each? How many in all possible ways?

10. How many combinations can be formed with 24 letters, taking 18 at a time?

11. How many different committees, consisting of 8 persons each, can be formed out of a corporation of 20 persons?

12. How many different numbers, of 4 different figures each, can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0?

13. How many different words, each consisting of 4 consonants and 2 vowels, can be formed from 8 consonants and 4 vowels?

The number of combinations of the 8 consonants, taken 4 at a time, is

$$\frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} = 70.$$

The number of combinations of the 4 vowels, taken 2 at a time, is

$$\frac{4 \cdot 3}{1 \cdot 2} = 6.$$

But any one of the 70 sets of consonants may be associated with any one of the 6 sets of vowels.

Hence there are in all 70×6, or 420 sets, each containing 4 consonants and 2 vowels.

Now each of these sets of 6 letters may have $\underline{|6}$, or 720, different permutations (Art. 448).

Therefore the whole number of different words is

$$420 \times 720 = 302400, \textit{ Ans.}$$

14. How many different words, each consisting of 3 consonants and 1 vowel, can be formed from 12 consonants and 3 vowels?

15. How many different committees, each consisting of 2 Republicans and 3 Democrats, can be formed from 14 Republicans and 21 Democrats?

16. Out of 9 red balls, 4 white balls, and 6 black balls, how many different combinations can be formed, each consisting of 5 red balls, 1 white ball, and 3 black balls?

17. How many different words, each consisting of 4 consonants and 3 vowels, can be formed from 10 consonants and 5 vowels?

18. Out of 11 physicians, 13 teachers, and 8 lawyers, how many different committees can be formed, each consisting of 3 physicians, 4 teachers, and 2 lawyers?

19. How many words of seven letters each can be formed from the letters a, b, c, d, e, f, g, each word being such that the letters a, b, c are never separated?

XXXIX. CONTINUED FRACTIONS.

453. A *continued fraction* is an expression of the form

$$a+\cfrac{b}{c+\cfrac{d}{e+\cdots}};$$

or, as it is usually written in practice,

$$a+\frac{b}{c+}\,\frac{d}{e+\cdots}.$$

We shall limit ourselves in the present chapter to continued fractions of the form

$$a+\frac{1}{b+}\,\frac{1}{c+\cdots},$$

where each numerator is unity, a is 0 or any positive integer, and each of the quantities b, c, ..., is a positive integer.

454. A *terminating* continued fraction is one in which the number of denominators is finite; as,

$$a+\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{d}.$$

It may be reduced to an ordinary fraction by the process of Art. 161.

An *infinite* continued fraction is one in which the number of denominators is indefinitely great.

455. In the continued fraction

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+\cdots},$$

a_1 is called the *first convergent;*

$a_1+\dfrac{1}{a_2}$ is called the *second convergent;*

$a_1+\dfrac{1}{a_2+}\,\dfrac{1}{a_3}$ is called the *third convergent;* and so on.

Note. If $a_1 = 0$, as in the continued fraction

$$\frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+\cdots},$$

then 0 is considered the first convergent.

456. *Any ordinary fraction in its lowest terms may be converted into a terminating continued fraction.*

Let the given fraction be $\frac{a}{b}$.

Divide a by b, and let a_1 denote the quotient and b_1 the remainder; then,

$$\frac{a}{b} = a_1 + \frac{b_1}{b} = a_1 + \cfrac{1}{\cfrac{b}{b_1}}.$$

Divide b by b_1, and let a_2 denote the quotient and b_2 the remainder; then,

$$\frac{a}{b} = a_1 + \cfrac{1}{a_2 + \cfrac{b_2}{b_1}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cfrac{b_1}{b_2}}}.$$

Again, divide b_1 by b_2, and let a_3 denote the quotient and b_3 the remainder; then,

$$\frac{a}{b} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{b_3}{b_2}}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cfrac{b_2}{b_3}}}}.$$

The process is the same as that of finding the Highest Common Factor of a and b (Art. 129); and since a and b are prime to each other, we must eventually obtain a remainder unity, at which point the operation terminates.

Hence any ordinary fraction in its lowest terms can be converted into a *terminating* continued fraction.

Example. Convert $\frac{62}{23}$ into a continued fraction.

$$\begin{array}{l}
23)62(2 = a_1 \\
\quad\; 46 \\
\quad\; \overline{16})23(1 = a_2 \\
\qquad\quad 16 \\
\qquad\quad \overline{7})16(2 = a_3 \\
\qquad\qquad 14 \\
\qquad\qquad \overline{2})7(3 = a_4 \\
\qquad\qquad\quad 6 \\
\qquad\qquad\quad \overline{1}
\end{array}$$

Therefore, $\frac{62}{23} = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2}$, *Ans.*

457. *A quadratic surd* (Art. 250) *may be converted into an infinite continued fraction.*

Example. Convert $\sqrt{6}$ into a continued fraction.

The greatest integer in $\sqrt{6}$ is 2; we then write

$$\sqrt{6} = 2 + (\sqrt{6} - 2).$$

Reducing $\sqrt{6} - 2$ to an equivalent fraction with a rational numerator (Art. 244), we have

$$\sqrt{6} = 2 + \frac{(\sqrt{6}-2)(\sqrt{6}+2)}{\sqrt{6}+2} = 2 + \frac{2}{\sqrt{6}+2}$$

$$= 2 + \frac{1}{\frac{\sqrt{6}+2}{2}}. \qquad (1)$$

The greatest integer in $\frac{\sqrt{6}+2}{2}$ is 2; we then write

$$\frac{\sqrt{6}+2}{2} = 2 + \frac{\sqrt{6}-2}{2}$$

$$= 2 + \frac{(\sqrt{6}-2)(\sqrt{6}+2)}{2(\sqrt{6}+2)} = 2 + \frac{1}{\sqrt{6}+2}.$$

Substituting in (1),

$$\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{\sqrt{6}+2}}. \qquad (2)$$

The greatest integer in $\sqrt{6}+2$ is 4; we then write

$$\sqrt{6}+2 = 4 + (\sqrt{6}-2) = 4 + \frac{(\sqrt{6}-2)(\sqrt{6}+2)}{\sqrt{6}+2}$$

$$= 4 + \frac{2}{\sqrt{6}+2} = 4 + \cfrac{1}{\cfrac{\sqrt{6}+2}{2}}.$$

Substituting in (2), we have

$$\sqrt{6} = 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{\cfrac{\sqrt{6}+2}{2}}}}.$$

The steps now recur, and we have

$$\sqrt{6} = 2 + \frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+\cdots}, \quad \textit{Ans.}$$

Note. An infinite continued fraction in which the elements recur, is called a *periodic* continued fraction.

458. *A periodic continued fraction may always be expressed as the root of a certain quadratic equation.*

Example. Express $\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+\cdots}$ as the root of a certain quadratic equation.

Let x denote the value of the fraction; then,

$$x = \frac{1}{1+}\,\frac{1}{3+x} = \frac{3+x}{3+x+1} = \frac{3+x}{4+x}.$$

Clearing of fractions,

$$4x + x^2 = 3 + x, \text{ or } x^2 + 3x = 3.$$

Solving the equation,

$$x = \frac{-3+\sqrt{9+12}}{2} = \frac{-3+\sqrt{21}}{2}, \quad \textit{Ans.}$$

Note. The + sign is taken before the radical, since x is evidently a positive quantity.

PROPERTIES OF CONVERGENTS.

459. Let the continued fraction be

$$a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n+}\,\frac{1}{a_{n+1}+\cdots};$$

and let p_r denote the numerator, and q_r the denominator, of the rth convergent (Art. 455) when expressed in its simplest form.

460. *To determine the law of formation of the successive convergents.*

The first convergent is a_1.

The second is $a_1 + \frac{1}{a_2} = \frac{a_1a_2+1}{a_2}$.

The third is $a_1 + \frac{1}{a_2+}\,\frac{1}{a_3} = a_1 + \frac{a_3}{a_2a_3+1} = \frac{a_1a_2a_3+a_1+a_3}{a_2a_3+1}$.

The third convergent may be written in the form

$$\frac{(a_1a_2+1)\,a_3+a_1}{a_2a_3+1};$$

in which we observe that

1. *The numerator is equal to the numerator of the preceding convergent, multiplied by the last denominator taken, plus the numerator of the convergent next but one preceding.*

2. *The denominator is equal to the denominator of the preceding convergent, multiplied by the last denominator taken, plus the denominator of the convergent next but one preceding.*

We will now prove by Induction (Note 1, Art. 363) that the above laws hold for all convergents after the second, when expressed in their simplest forms.

Assume that the laws hold for all convergents as far as the nth.

The nth convergent is

$$\frac{p_n}{q_n}=a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n}.$$

Then since the last denominator is a_n, we have

$$p_n=a_np_{n-1}+p_{n-2},\text{ and } q_n=a_nq_{n-1}+q_{n-2}. \tag{1}$$

Whence,
$$\frac{p_n}{q_n}=\frac{a_np_{n-1}+p_{n-2}}{a_nq_{n-1}+q_{n-2}}. \tag{2}$$

The $(n+1)$st convergent is

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n+}\,\frac{1}{a_{n+1}},$$

which differs from the nth only in having $a_n+\dfrac{1}{a_{n+1}}$, or $\dfrac{a_na_{n+1}+1}{a_{n+1}}$, in place of a_n.

Substituting $\dfrac{a_na_{n+1}+1}{a_{n+1}}$ for a_n in (2), we have

$$\frac{p_{n+1}}{q_{n+1}}=\frac{\dfrac{a_na_{n+1}+1}{a_{n+1}}p_{n-1}+p_{n-2}}{\dfrac{a_na_{n+1}+1}{a_{n+1}}q_{n-1}+q_{n-2}}$$

$$=\frac{a_{n+1}(a_np_{n-1}+p_{n-2})+p_{n-1}}{a_{n+1}(a_nq_{n-1}+q_{n-2})+q_{n-1}}$$

$$=\frac{a_{n+1}p_n+p_{n-1}}{a_{n+1}q_n+q_{n-1}},\text{ by (1)}. \tag{3}$$

It is evident that the second member of (3) is the simplest form of the $(n+1)$st convergent, and therefore

$$p_{n+1}=a_{n+1}p_n+p_{n-1},\text{ and } q_{n+1}=a_{n+1}q_n+q_{n-1}.$$

These results are in accordance with the laws stated on the preceding page.

Hence, if the laws hold for all convergents as far as the nth, they also hold as far as the $(n+1)$st.

But we know that they hold as far as the third convergent, and hence they also hold as far as the fourth; and since they hold as far as the fourth, they also hold as far as the fifth; and so on.

Therefore the laws hold for all convergents after the second.

Example. Find the first five convergents of

$$1+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+\cdots}.$$

The first convergent is 1, and the second is $1+1$, or 2.

Then by aid of the laws just proved,

$$\text{the third is } \frac{2\cdot 2+1}{1\cdot 2+1}=\frac{5}{3};$$

$$\text{the fourth is } \frac{5\cdot 3+2}{3\cdot 3+1}=\frac{17}{10};$$

$$\text{the fifth is } \frac{17\cdot 4+5}{10\cdot 4+3}=\frac{73}{43}.$$

461. *The difference between two consecutive convergents* $\frac{p_n}{q_n}$ *and* $\frac{p_{n+1}}{q_{n+1}}$ *is equal to* $\frac{1}{q_n q_{n+1}}$.

The difference between the first and second convergents is

$$\left(a_1+\frac{1}{a_2}\right)-a_1=\frac{1}{a_2}.$$

Thus the theorem holds for the first and second convergents.

Assume that it holds for the nth and $(n+1)$st convergents; that is,

$$\frac{p_n}{q_n}\sim\frac{p_{n+1}}{q_{n+1}}=\frac{1}{q_n q_{n+1}}, \text{ or } p_n q_{n+1}\sim p_{n+1}q_n=1. \qquad (1)$$

Then,

$$\frac{p_{n+1}}{q_{n+1}}\sim\frac{p_{n+2}}{q_{n+2}}=\frac{p_{n+1}}{q_{n+1}}\sim\frac{a_{n+2}p_{n+1}+p_n}{a_{n+2}q_{n+1}+q_n} \text{ (Art. 460)}$$

$$=\frac{(a_{n+2}p_{n+1}q_{n+1}+p_{n+1}q_n)\sim(a_{n+2}p_{n+1}q_{n+1}+p_nq_{n+1})}{q_{n+1}(a_{n+2}q_{n+1}+q_n)}$$

$$=\frac{p_{n+1}q_n\sim p_nq_{n+1}}{q_{n+1}q_{n+2}}\text{ (Art. 460)}=\frac{1}{q_{n+1}q_{n+2}},\text{ by (1).}$$

Hence if the theorem holds for any pair of consecutive convergents, it also holds for the next pair.

But we know that it holds for the first and second convergents, and hence it also holds for the second and third; and since it holds for the second and third, it also holds for the third and fourth; and so on.

Therefore the theorem holds universally.

462. It follows from Art. 461 that p_n and q_n can have no common divisor except unity; for if they had, it would be a divisor of $p_nq_{n+1}\sim p_{n+1}q_n$, or unity, which is impossible.

Therefore all convergents formed in accordance with the laws of Art. 460 are in their *lowest terms*.

463. *The even convergents are greater, and the odd convergents less, than the fraction itself.*

I. The first convergent, a_1, is *less* than the fraction itself, since $\frac{1}{a_2+\cdots}$ is omitted.

II. The second, $a_1+\frac{1}{a_2}$, is *greater*, because its denominator a_2 is less than $a_2+\frac{1}{a_3+\cdots}$, the denominator of the fraction.

III. The third, $a_1+\frac{1}{a_2+}\frac{1}{a_3}$, is *less*, because, by II., the denominator $a_2+\frac{1}{a_3}$ is greater than $a_2+\frac{1}{a_3+}\frac{1}{a_4+\cdots}$, the denominator of the fraction; and so on.

Hence the first, third, ..., convergents are less, and the second, fourth, ..., convergents greater than the fraction itself.

464. *Any convergent is nearer than the preceding convergent to the value of the fraction itself.*

By Art. 460, $$\frac{p_{n+2}}{q_{n+2}}=\frac{a_{n+2}p_{n+1}+p_n}{a_{n+2}q_{n+1}+q_n}.$$

The fraction itself is obtained from its $(n+2)$nd convergent by putting $a_{n+2}+\frac{1}{a_{n+3}+\cdots}$ in place of a_{n+2}.

Hence, denoting the value of the fraction itself by x, we have

$$x=\frac{\left[a_{n+2}+\dfrac{1}{a_{n+3}+\cdots}\right]p_{n+1}+p_n}{\left[a_{n+2}+\dfrac{1}{a_{n+3}+\cdots}\right]q_{n+1}+q_n}=\frac{mp_{n+1}+p_n}{mq_{n+1}+q_n},$$

where m stands for $a_{n+2}+\frac{1}{a_{n+3}+\cdots}$.

Now, $$x\sim\frac{p_n}{q_n}=\frac{mp_{n+1}+p_n}{mq_{n+1}+q_n}\sim\frac{p_n}{q_n}$$

$$=\frac{m(p_{n+1}q_n\sim p_nq_{n+1})}{q_n(mq_{n+1}+q_n)}$$

$$=\frac{m}{q_n(mq_{n+1}+q_n)}\ \text{(Art. 461).} \qquad (1)$$

Also, $$x\sim\frac{p_{n+1}}{q_{n+1}}=\frac{mp_{n+1}+p_n}{mq_{n+1}+q_n}\sim\frac{p_{n+1}}{q_{n+1}}$$

$$=\frac{p_nq_{n+1}\sim p_{n+1}q_n}{q_{n+1}(mq_{n+1}+q_n)}=\frac{1}{q_{n+1}(mq_{n+1}+q_n)}. \qquad (2)$$

Since a_{n+2} is a positive integer, $a_{n+2}+\frac{1}{a_{n+3}+\cdots}$ is >1; that is, m is >1.

And since $q_{n+1}=a_{n+1}q_n+q_{n-1}$ (Art. 460), q_{n+1} is $>q_n$.

Therefore the fraction (2) is less than the fraction (1), for it has a smaller numerator and a greater denominator.

Hence the $(n+1)$st convergent is nearer than the nth to the value of the fraction itself.

465. By Art. 464, the difference between the fraction itself and its nth convergent is

$$\frac{m}{q_n(mq_{n+1}+q_n)}, \text{ or } \frac{1}{q_n\left(q_{n+1}+\dfrac{q_n}{m}\right)}. \qquad (1)$$

Since m is >1 (Art. 464), the denominator $q_n\left(q_{n+1}+\dfrac{q_n}{m}\right)$ is $< q_n(q_{n+1}+q_n)$.

The denominator is also $> q_n q_{n+1}$.

Hence the fraction (1) is $> \dfrac{1}{q_n(q_{n+1}+q_n)}$, and $< \dfrac{1}{q_n q_{n+1}}$.

That is, the *error* made in taking the nth convergent for the fraction itself lies between the limits

$$\frac{1}{q_n(q_{n+1}+q_n)} \text{ and } \frac{1}{q_n q_{n+1}}.$$

EXAMPLES.

466. Convert each of the following into a continued fraction, and find in each case the first five convergents:

1. $\dfrac{53}{39}$. **3.** $3.\dot{6}1$. **5.** $\dfrac{749}{326}$. **7.** $\dfrac{436}{345}$.

2. $\dfrac{72}{91}$. **4.** $\dfrac{112}{153}$. **6.** $\dfrac{144}{89}$. **8.** $\dfrac{3015}{6961}$.

Convert each of the following into a continued fraction, find in each case the first four convergents, and determine limits to the error made in taking the fourth convergent for the fraction itself:

9. $\sqrt{5}$. **10.** $\sqrt{3}$. **11.** $\sqrt{11}$. **12.** $\sqrt{7}$.

Express each of the following in the form of a surd:

13. $\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{3+\cdots}$. **15.** $2+\dfrac{1}{1+}\,\dfrac{1}{1+\cdots}$.

14. $\dfrac{1}{1+}\,\dfrac{1}{4+}\,\dfrac{1}{1+}\,\dfrac{1}{4+\cdots}$. **16.** $3+\dfrac{1}{1+}\,\dfrac{1}{8+}\,\dfrac{1}{1+}\,\dfrac{1}{8+\cdots}$.

17. The ratio of the circumference of a circle to its diameter is approximately equal to 3.14159; express this decimal as a continued fraction, and find the first four convergents.

18. The modulus of the common system of logarithms is approximately equal to .43429; express this decimal as a continued fraction, find its seventh convergent, and determine limits to the error made in taking this convergent for the fraction itself.

19. The base of the Napierian system of logarithms is 2.7183 approximately; express this decimal as a continued fraction, find its eighth convergent, and determine limits to the error made in taking this convergent for the fraction itself.

20. Express the positive root of the equation

$$x^2 - x - 11 = 0$$

as a continued fraction, and find the first five convergents.

XL. GENERAL THEORY OF EQUATIONS.

467. Every equation of the nth degree (Art. 167), involving one unknown quantity, can be written in the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0; \qquad (1)$$

where the coefficients $p_1, p_2, \cdots, p_n$ may be positive or negative, integral or fractional, rational or irrational, real or imaginary, or zero.

If none of the coefficients $p_1, p_2, \cdots, p_n$ are zero, the equation is said to be *Complete;* if one or more of them are zero, the equation is said to be *Incomplete.*

We shall hereafter speak of (1) as the *General Form* of an equation of the nth degree.

468. We assume that every equation of the above form has at least one root (Art. 169), real or imaginary.

469. Divisibility of Equations.

If a is a root of the equation

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0,$$

then the first member is divisible by $x - a$.

The division of the first member by $x - a$ may be carried out until a remainder is obtained which does not contain x.

Let Q denote the quotient, and R the remainder.

Then the given equation may be made to take the form

$$(x - a)Q + R = 0. \qquad (1)$$

Since a is a root of the given equation, equation (1) must be satisfied when x is put equal to a.

Putting $x = a$, we have, since R does not contain x,

$$R = 0.$$

Therefore, $x - a$ is a factor of the first member of the given equation, for it is contained in it without a remainder.

470. *Conversely, if the first member of*

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0$$

is divisible by $x - a$, then a is a root of the equation.

For, since the first member of the given equation is divisible by $x - a$, the equation may be made to take the form

$$(x - a)Q = 0;$$

and it follows from Art. 282 that a is a root of this equation.

471. It follows from Art. 470 that if the first member of

$$p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0$$

is divisible by $ax + b$, then $-\frac{b}{a}$ is a root of the equation.

EXAMPLES.

472. Prove, by the method of Art. 470 or Art. 471,

1. That 5 is a root of $x^3 - 2x^2 - 19x + 20 = 0$.
2. That -2 is a root of $x^4 - 3x^2 + 4x + 4 = 0$.
3. That -3 is a root of $2x^3 + 3x^2 - 2x + 21 = 0$.
4. That $\frac{2}{3}$ is a root of $3x^4 - 8x^3 + 13x^2 - 9x + 2 = 0$.
5. That -4 is not a root of $x^4 - x^3 + 7x - 12 = 0$.
6. That $-\frac{1}{4}$ is a root of $8x^4 + 6x^3 - 15x^2 - 16x - 3 = 0$.
7. That $\frac{2}{5}$ is a root of $125x^3 - 8 = 0$.
8. That $\frac{3}{4}$ is not a root of $16x^3 + 8x^2 - 23x - 3 = 0$.

473. Number of Roots.

An equation of the nth degree cannot have more than n different roots.

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0. \qquad (1)$$

By Art. 468, equation (1) must have at least one root.

Let a be this root; then the first member is divisible by $x-a$ (Art. 469), and the equation may be put in the form

$$(x-a)(x^{n-1}+q_2x^{n-2}+\cdots+q_{n-1}x+q_n)=0.$$

By Art. 282, this equation may be solved by placing

$$x-a=0,$$

and $$x^{n-1}+q_2x^{n-2}+\cdots+q_{n-1}x+q_n=0. \tag{2}$$

Equation (2) must also have at least one root.

Let b be this root; then (2) may be written

$$(x-b)(x^{n-2}+r_3x^{n-3}+\cdots+r_{n-1}x+r_n)=0.$$

Whence, $$x-b=0,$$

and $$x^{n-2}+r_3x^{n-3}+\cdots+r_{n-1}x+r_n=0.$$

Continuing the above process until $n-1$ binomial factors have been divided out, we shall arrive finally at an equation of the *first* degree,

$$x-k=0, \text{ whence } x=k.$$

Then the given equation has the n roots $a, b, \cdots, k$.

Note. It should be observed that the roots are not necessarily *unequal;* thus, the equation $x^3-3x^2+4=0$ can be written in the form $(x+1)(x-2)(x-2)=0$, and its three roots are -1, 2, and 2.

474. The principle of Art. 473 is usually stated as follows:

An equation of the nth degree has n roots;

which means that it may have n different roots, but cannot have more than n.

475. Depression of Equations.

It follows from Art. 473 that, if m roots of an equation of the nth degree are known, the equation may be depressed to another equation of the $(n-m)$th degree, which shall contain the other $n-m$ roots.

Thus, if all but two of the roots of an equation are known, these two may be obtained from the depressed equation by the rules for quadratics.

1. Two roots of the equation $9x^4 - 37x^2 - 8x + 20 = 0$ are 2 and $-\frac{5}{3}$; what are the others?

By Art. 469, the first member of the given equation is divisible by $(x-2)(3x+5)$, or $3x^2 - x - 10$.

Dividing $9x^4 - 37x^2 - 8x + 20$ by $3x^2 - x - 10$, the quotient is $3x^2 + x - 2$.

Then the depressed equation is $3x^2 + x - 2 = 0$.

Solving this by the rules for quadratics, $x = \frac{2}{3}$ or -1.

EXAMPLES.

2. One root of $x^3 - 37x + 84 = 0$ is 3; find the others.

3. One root of $2x^3 + 5x^2 - 43x - 90 = 0$ is -2; what are the others?

4. One root of $24x^3 - 46x^2 + 29x - 6 = 0$ is $\frac{1}{2}$; what are the others?

5. One root of $32x^3 - 32x^2 - 94x + 39 = 0$ is $-\frac{3}{2}$; what are the others?

6. Two roots of $6x^4 + 7x^3 - 38x^2 - 25x + 50 = 0$ are 1 and 2; what are the others?

7. One root of $15x^3 + x^2 - 31x + 15 = 0$ is $-\frac{5}{3}$; what are the others?

8. Two roots of $60x^3 + 17x^2 - 29x - 6 = 0$ are $\frac{2}{3}$ and $-\frac{3}{4}$; what is the other?

9. Two roots of $36x^4 - 445x^2 + 49 = 0$ are $\frac{1}{3}$ and $-\frac{7}{2}$; what are the others?

10. Two roots of $x^4 + 10ax^3 + 35a^2x^2 + 50a^3x + 24a^4 = 0$ are $-a$ and $-3a$; what are the others?

11. One root of

$$x^3 - (m+2)x^2 - (m^2+4m+5)x + m^3 + 6m^2 + 11m + 6 = 0$$

is $m+1$; what are the others?

476. Formation of Equations.

It follows from Art. 473 that if the roots of the equation

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0$$

are a, b, $\cdots$, k, the equation can be written in the form

$$(x-a)(x-b)\cdots(x-k) = 0.$$

Hence, to form an equation which shall have any required roots,

Subtract each root from x, and place the product of the resulting expressions equal to zero. (Compare Art. 280.)

1. Form an equation having the roots 1, $\frac{1}{2}$, and $-\frac{5}{3}$.

By the rule, $(x-1)(x-\frac{1}{2})(x+\frac{5}{3}) = 0.$

Multiplying the terms of the second and third factors by 2 and 3, respectively, we have

$$(x-1)(2x-1)(3x+5) = 0.$$

Expanding, $6x^3 + x^2 - 12x + 5 = 0.$

EXAMPLES.

Form the equations whose roots shall be:

2. 1, 2, 3.

3. -2, -3, 5.

4. 1, 4, -5, 0.

5. $-\frac{1}{2}$, $-\frac{2}{3}$, $-\frac{3}{4}$.

6. 6, -1, $1\frac{1}{2}$, $-\frac{1}{3}$.

7. -2, -2, $1\frac{2}{3}$, $1\frac{2}{3}$.

8. 4, -3, $-\frac{1}{4}$, $\frac{1}{3}$.

9. $2 \pm \sqrt{3}$, $-2 \pm \sqrt{3}$.

10. $\frac{1}{2}(1 \pm 2\sqrt{5})$, $\frac{1}{2}(-2 \pm \sqrt{5})$.

477. Composition of Coefficients.

By Art. 476, the equation of the nth degree whose roots are a, b, c, d, $\cdots$, k, l, m, is

$$(x-a)(x-b)(x-c)(x-d)\cdots(x-m) = 0. \qquad (1)$$

By actual multiplication, we obtain

$$(x-a)(x-b)=x^2-(a+b)x+ab;$$

$$(x-a)(x-b)(x-c)$$
$$=x^3-(a+b+c)x^2+(ab+bc+ca)x-abc;$$

and so on.

When all the factors of the first member of (1) have been multiplied together, the result will be in the form

$$x^n+p_1x^{n-1}+p_2x^{n-2}+p_3x^{n-3}+\cdots+p_n;$$

where $p_1=-(a+b+c+\cdots+k+l+m);$

$$p_2=ab+ac+bc+\cdots+lm;$$

$$p_3=-(abc+abd+acd+\cdots+klm);$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$p_n=\pm\, abcd\cdots klm$, according as n is even or odd.

Hence, in an equation of the nth degree in the general form,

The coefficient of the second term is equal to minus the sum of all the roots.

The coefficient of the third term is equal to the sum of the products of the roots, taken two at a time.

The coefficient of the fourth term is equal to minus the sum of the products of the roots, taken three at a time; etc.

The last term is equal to plus or minus the product of all the roots, according as n is even or odd.

478. It follows from Art. 477 that, if an equation of the nth degree is in the general form,

If the second term is wanting, the sum of the roots is 0.

If the last term is wanting, at least one root is 0.

If the last term is an integer, it is divisible by every integral root.

479. If all but one of the roots of an equation of the nth degree in the general form are known, the remaining root may be found by adding the sum of the known roots to the coefficient of the second term of the given equation, and changing the sign of the result; or, by dividing the last term of the given equation by plus or minus the product of the known roots according as n is even or odd.

If all but two of the roots are known, the coefficient of the second term of the depressed equation may be found by adding the sum of the known roots to the coefficient of the second term of the given equation; and the last term of the depressed equation may be found by dividing the last term of the given equation by plus or minus the product of the known roots according as n is even or odd.

1. Two roots of $x^4+2x^3-13x^2-14x+24=0$ are 1 and -4; what are the others?

The sum of the known roots is -3, and their product is -4.

Then the coefficient of the second term of the depressed equation is $-3+2$, or -1; and the last term is $24 \div (-4)$, or -6.

Then the depressed equation is $x^2-x-6=0$.

Solving this by the rules for quadratics, $x=3$, or -2.

EXAMPLES.

In each of the following, find the sum of the roots and the product of the roots:

2. $2x^4-13x^3-91x^2+390x+216=0$.

3. $5x^5+8x^4+29x^3-109x-68=0$.

4. $4x^3-7x+21=0$.

In each of the following, obtain the roots by the rules of Art. 479:

5. Two roots of $x^3-4x^2-17x+60=0$ are -4 and 5; what is the other?

6. Three roots of $x^4 - 45x^2 + 40x + 84 = 0$ are 2, 6, and -7; what is the other?

7. Four roots of $x^5 - 4x^4 - 5x^3 + 20x^2 + 4x - 16 = 0$ are 1, -1, -2, and 4; what is the other?

8. Two roots of $x^4 + 2x^3 - 13x^2 - 38x - 24 = 0$ are -1 and 4; what are the others?

9. Three roots of $x^5 - 74x^3 - 24x^2 + 937x - 840 = 0$ are 1, -7, and 8; what are the others?

480. Fractional Roots.

An equation in the general form, whose coefficients are integral, cannot have as a root a rational fraction in its lowest terms.

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0, \qquad (1)$$

where the coefficients $p_1, p_2, \cdots, p_n$ are integral.

If possible, let $\frac{a}{b}$, a rational fraction in its lowest terms, be a root of (1); then,

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + p_2\left(\frac{a}{b}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0.$$

Multiplying each term by b^{n-1}, and transposing,

$$\frac{a^n}{b} = -(p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1}).$$

Since, by hypothesis, $\frac{a}{b}$ is in its lowest terms, a and b have no common divisor.

We then have a rational fraction in its lowest terms equal to an integral expression, which is impossible.

Therefore, the given equation cannot have as a root a rational fraction in its lowest terms.

481. The imaginary quantities $a+b\sqrt{-1}$ and $a-b\sqrt{-1}$ are said to be **Conjugate**.

482. Imaginary Roots.

If an imaginary quantity is a root of an equation in the general form with real coefficients, its conjugate (Art. 481) *is also a root.*

Let the equation be

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0, \tag{1}$$

where the coefficients $p_1, \cdots, p_n$ are real.

Let $a + b\sqrt{-1}$ be a root of (1); then,

$$(a + b\sqrt{-1})^n + p_1(a + b\sqrt{-1})^{n-1} + \cdots \\ + p_{n-1}(a + b\sqrt{-1}) + p_n = 0.$$

Expanding each term by the Binomial Theorem, we have by Art. 248,

$$a^n + na^{n-1}b\sqrt{-1} - \frac{n(n-1)}{\underline{|2}}a^{n-2}b^2 \\ - \frac{n(n-1)(n-2)}{\underline{|3}}a^{n-3}b^3\sqrt{-1} + \cdots \\ + p_1\left[a^{n-1} + (n-1)a^{n-2}b\sqrt{-1} - \frac{(n-1)(n-2)}{\underline{|2}}a^{n-3}b^2 - \cdots\right] \\ + \cdots + p_{n-1}(a + b\sqrt{-1}) + p_n = 0. \tag{2}$$

Collecting the real and imaginary terms, we shall have a result of the form

$$P + Q\sqrt{-1} = 0,$$

where P and Q are real.

In order that this equation may hold, we must have

$$P = 0, \text{ and } Q = 0.$$

Now substituting $a - b\sqrt{-1}$ for x in the first member of equation (1), it becomes

$$(a - b\sqrt{-1})^n + p_1(a - b\sqrt{-1})^{n-1} + \cdots \\ + p_{n-1}(a - b\sqrt{-1}) + p_n. \tag{3}$$

Expanding each term by the Binomial Theorem, we shall have a result which differs from the first member of equation (2) only in having the second, fourth, sixth, etc., terms of each expansion, or those involving $\sqrt{-1}$ as a factor, *changed in sign.*

Then, collecting the real and imaginary terms, the expression (3) is equal to

$$P - Q\sqrt{-1},$$

where P and Q have the same meanings as before.

But since $P = 0$ and $Q = 0$, we have $P - Q\sqrt{-1} = 0$.

Whence, $a - b\sqrt{-1}$ is a root of equation (1).

483. The product of the factors of the first member of equation (1), Art. 482, corresponding to the conjugate imaginary roots $a + b\sqrt{-1}$ and $a - b\sqrt{-1}$, is

$$\begin{aligned}&[x - (a + b\sqrt{-1})][x - (a - b\sqrt{-1})] \qquad \text{(Art. 469)}\\ &= (x - a - b\sqrt{-1})(x - a + b\sqrt{-1})\\ &= (x - a)^2 - (b\sqrt{-1})^2 = (x - a)^2 + b^2;\end{aligned}$$

and is therefore positive for every real value of x.

484. It follows from Arts. 473 and 482 that every equation of odd degree has at least one real root; for an equation cannot have an odd number of imaginary roots.

TRANSFORMATION OF EQUATIONS.

485. *To transform an equation into another which shall have the same roots with contrary signs.*

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0. \qquad (1)$$

Substituting $-y$ for x, we have

$$(-y)^n + p_1(-y)^{n-1} + p_2(-y)^{n-2} + \cdots + p_{n-1}(-y) + p_n = 0.$$

Dividing each term by $(-1)^n$, we have

$$y^n + p_1\frac{y^{n-1}}{(-1)} + p_2\frac{y^{n-2}}{(-1)^2} + \cdots + p_{n-1}\frac{y}{(-1)^{n-1}} + \frac{p_n}{(-1)^n} = 0,$$

or, $$y^n - p_1y^{n-1} + p_2y^{n-2} - \cdots \pm p_{n-1}y \mp p_n = 0; \qquad (2)$$

the upper or lower signs being taken according as n is odd or even.

It follows from (1) and (2) that the desired transformation may be effected by simply *changing the signs of the alternate terms commencing with the second.*

Note. If the equation is *incomplete*, any missing term must be supplied with the coefficient zero before applying the rule.

1. Transform the equation $x^3 - 10x + 4 = 0$ into another which shall have the same roots with contrary signs.

The equation may be written in the form

$$x^3 + 0 \cdot x^2 - 10x + 4 = 0.$$

Then, by the rule, the transformed equation is

$$x^3 - 0 \cdot x^2 - 10x - 4 = 0, \text{ or } x^3 - 10x - 4 = 0.$$

EXAMPLES.

Transform each of the following into an equation which shall have the same roots with contrary signs:

2. $x^4 + 3x^3 - 2x^2 - 5x + 7 = 0.$ **3.** $x^5 - 5x^2 + 16 = 0.$

4. $x^7 - 4x^6 - 6x^3 + 12x^2 - 25 = 0.$

486. *To transform an equation into another whose roots shall be m times those of the first.*

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

Substituting $\frac{y}{m}$ for x, whence $y = mx$, we have

$$\left(\frac{y}{m}\right)^n + p_1\left(\frac{y}{m}\right)^{n-1} + p_2\left(\frac{y}{m}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{y}{m}\right) + p_n = 0.$$

Multiplying each term by m^n,

$$y^n + p_1 m y^{n-1} + p_2 m^2 y^{n-2} + \cdots + p_{n-1} m^{n-1} y + p_n m^n = 0.$$

Hence, to effect the desired transformation, *multiply the second term by m, the third term by m^2, and so on.*

1. Transform the equation $x^3 + 7x^2 - 6 = 0$ into another whose roots shall be 4 times those of the first.

The equation may be written $x^3 + 7x^2 + 0 \cdot x - 6 = 0$.

Then, by the rule, the transformed equation is

$$x^3 + 4 \cdot 7x^2 + 4^2 \cdot 0 \cdot x - 4^3 \cdot 6 = 0, \text{ or } x^3 + 28x^2 - 384 = 0.$$

EXAMPLES.

2. Transform $x^3 - 5x^2 - 7x + 11 = 0$ into an equation whose roots shall be those of the first multiplied by 3.

3. Transform $x^4 + 6x^3 - 2x - 5 = 0$ into an equation whose roots shall be those of the first multiplied by -5.

4. Transform $2x^3 - 5x + 7 = 0$ into an equation whose roots shall be those of the first multiplied by $\frac{2}{3}$.

5. Transform $6x^4 - 3x^3 + 8x^2 - 16 = 0$ into an equation whose roots shall be those of the first multiplied by $-\frac{5}{4}$.

487. *To transform an equation with fractional coefficients into another whose coefficients shall be integral, that of the first term being unity.*

This may be effected by multiplying the roots of the equation by m (Art. 486), and then giving m such a value as will make all the coefficients integral.

By giving m the *least* value which will make all the coefficients integral, the result will be obtained in its simplest form.

1. Transform $x^3 - \dfrac{x^2}{3} - \dfrac{x}{36} + \dfrac{1}{108} = 0$ into an equation with integral coefficients, that of the first term being unity.

Multiplying the roots by m, we have

$$x^3 - \frac{m}{3}x^2 - \frac{m^2}{36}x + \frac{m^3}{108} = 0.$$

It is evident by inspection that the least value of m which will make all the coefficients integral is 6.

Putting $m = 6$, we obtain

$$x^3 - 2x^2 - x + 2 = 0;$$

whose roots are 6 times those of the given equation.

EXAMPLES.

Transform each of the following into an equation with integral coefficients, that of the first term being unity:

2. $x^3 + \frac{3x^2}{2} + \frac{x}{8} - 1 = 0.$

3. $x^3 - \frac{2x}{5} + \frac{13}{20} = 0.$

4. $x^3 + \frac{3x^2}{7} - \frac{15}{56} = 0.$

5. $x^4 + \frac{5x^3}{3} - \frac{4x^2}{27} - \frac{1}{72} = 0.$

6. $x^4 - \frac{4x^3}{3} + \frac{5x}{81} - \frac{1}{27} = 0.$

7. $x^4 - \frac{7x^2}{45} - \frac{2x}{75} + \frac{2}{375} = 0.$

488. *To transform an equation into another whose roots shall be those of the first increased by m.*

Let the equation be

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0. \qquad (1)$$

Substituting $y - m$ for x, whence $y = x + m$, we have

$$(y-m)^n + p_1(y-m)^{n-1} + \cdots + p_{n-1}(y-m) + p_n = 0. \qquad (2)$$

Expanding the powers of $y - m$ by the Binomial Theorem, and collecting the terms involving like powers of y, we shall have a result of the form

$$y^n + q_1y^{n-1} + \cdots + q_{n-1}y + q_n = 0, \qquad (3)$$

whose roots are those of the given equation increased by m.

489. If m and the coefficients of the given equation are integers, the coefficients of the transformed equation may be conveniently found by the following method.

Putting $x+m$ for y in (3), we obtain

$$(x+m)^n + q_1(x+m)^{n-1} + \cdots + q_{n-1}(x+m) + q_n = 0. \quad (4)$$

Equation (4) must evidently take the same form as (1) on expanding the powers of $x+m$, and collecting the terms involving like powers of x.

Dividing the first member of (4) by $x+m$, we have

$$(x+m)^{n-1} + q_1(x+m)^{n-2} + \cdots + q_{n-2}(x+m) + q_{n-1} \quad (5)$$

as a quotient, and q_n as a remainder.

Dividing (5) by $x+m$, we have

$$(x+m)^{n-2} + q_1(x+m)^{n-3} + \cdots + q_{n-3}(x+m) + q_{n-2}$$

as a quotient, and q_{n-1} as a remainder; and so on.

Hence, to find the coefficients of the transformed equation,

Divide the first member of the given equation by $x+m$*; the remainder will be the last term of the transformed equation.*

Divide the quotient just found by $x+m$*; the remainder will be the coefficient of the next to the last term of the transformed equation; and so on.*

490. To transform an equation into another whose roots shall be those of the first *diminished* by m, we change $y-m$ to $y+m$ in the method of Art. 488, and $x+m$ to $x-m$ in the rule of Art. 489.

EXAMPLES.

491. 1. Transform $x^3 - 7x + 6 = 0$ into an equation whose roots shall be those of the first increased by 2.

We may either substitute $y-2$ for x in the given equation, or use the rule of Art. 489.

In the latter case, dividing $x^3 - 7x + 6$ by $x+2$, we have $x^2 - 2x - 3$ as a quotient, and 12 as a remainder.

Dividing $x^2 - 2x - 3$ by $x + 2$, we have $x - 4$ as a quotient, and 5 as a remainder.

Dividing $x - 4$ by $x + 2$, we have -6 as a remainder.

Then the transformed equation is

$$x^3 - 6x^2 + 5x + 12 = 0, \text{ Ans.}$$

2. Transform $x^3 + 2x^2 - 7x - 72 = 0$ into an equation whose roots shall be less by 4.

3. Transform $x^3 - 5x^2 + 4x - 23 = 0$ into an equation whose roots shall be greater by 5.

4. Transform $x^4 - x^3 - 2x^2 + 7x - 81 = 0$ into an equation whose roots shall be greater by 3.

5. Transform $x^4 + 3x^2 - 5x + 2 = 0$ into an equation whose roots shall be less by 6.

492. *To transform the equation*

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0,$$

where p_1 is not zero, into another whose second term shall be wanting.

Expanding the powers of $y - m$ in the first member of (2), Art. 488, and collecting the terms involving like powers of y, we have

$$y^n + (p_1 - mn)y^{n-1} + \cdots = 0.$$

If m is so taken that $p_1 - mn = 0$, whence $m = \frac{p_1}{n}$, the coefficient of y^{n-1} will be zero.

Hence, the desired transformation may be effected by *putting x equal to y, minus the coefficient of the second term of the given equation divided by the degree of the equation.*

1. Transform $x^3 - 6x^2 + 9x - 6 = 0$ into an equation whose second term shall be wanting.

Putting $x = y - \frac{-6}{3} = y + 2$, we have

$$(y+2)^3 - 6(y+2)^2 + 9(y+2) - 6 = 0,$$

or, $y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 9y + 18 - 6 = 0,$

or, $$y^3 - 3y - 4 = 0;$$

whose roots are those of the given equation diminished by 2.

EXAMPLES.

Transform each of the following into an equation whose second term shall be wanting:

2. $x^3 + 9x^2 - 3x + 5 = 0.$
3. $x^3 - x^2 - 4 = 0.$
4. $x^4 - 8x^3 - 5x + 1 = 0.$
5. $x^5 + 5x^4 - 9x^2 - 28 = 0.$

DESCARTES' RULE OF SIGNS.

493. If an equation of the nth degree is in the general form (Art. 467), a *Permanence* of sign occurs when two successive terms have the *same* sign, and a *Variation* of sign occurs when two successive terms have *opposite* signs.

Thus, in the equation $x^6 - 3x^4 - x^3 + 5x + 1 = 0$, there are two permanences and two variations.

494. Descartes' Rule of Signs.

No equation, whether complete or incomplete, can have a greater number of positive roots than it has variations of sign; and no complete equation can have a greater number of negative roots than it has permanences of sign.

Let an equation in the general form have the following signs,

$$+ + 0 - + 0\ 0 - - + 0\ 0\ 0 - + + 0 +$$

the missing terms being supplied with zero coefficients.

If we introduce a new positive root a, we multiply this by $x - a$ (Art. 476).

Writing only the *signs* which occur in the process, we have

$$\begin{array}{ccccccccccccccccccccl}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & & \\
+ & + & 0 & \dot{-} & \dot{+} & 0 & 0 & \dot{-} & - & \dot{+} & 0 & 0 & 0 & \dot{-} & \dot{+} & + & 0 & + & & (1) \\
+ & - & & & & & & & & & & & & & & & & & & \\
\hline
+ & + & 0 & - & + & 0 & 0 & - & - & + & 0 & 0 & 0 & - & + & + & 0 & + & & \\
 & - & - & 0 & + & - & 0 & 0 & + & + & - & 0 & 0 & 0 & + & - & - & 0 & - & \\
\hline
+ & m & - & - & + & - & 0 & - & m & + & - & 0 & 0 & - & + & m & - & + & - & (2) \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 &
\end{array}$$

where m signifies a term which may be $+$, 0, or $-$.

Now, in (1), let a dot be placed over the first minus sign, then over the next plus sign, then over the next minus sign, and so on.

The number of dots shows the number of variations; thus, in (1) there are six variations.

In the above result, we observe the following laws:

I. Directly under each dotted term of (1) is a term of (2) *having the same sign.*

Thus, the terms numbered 4, 5, 8, 10, 14, and 15, in (1) and (2), have the same sign.

II. The last term of (2) is of opposite sign to the term directly under the last dotted term of (1).

The above laws are easily seen to hold universally.

By the first law, however the terms marked m are taken, there are at least as many variations in the first fifteen terms of (2) as in (1); and by the second law, there is at least one variation in the remaining terms of (2).

Hence, the introduction of a new positive root increases the number of variations in the equation by at least one.

If, then, we form the product of all the factors corresponding to the negative and imaginary roots of an equation, multiplying the result by the factor corresponding to each positive root introduces at least one variation.

Hence, the equation cannot have a greater number of positive roots than it has variations of sign.

To prove the second part of Descartes' Rule, let $-y$ be substituted for x in any *complete* equation.

Then since the signs of the alternate terms commencing with the second are changed (Art. 485), the original *permanences* of sign become *variations.*

But the transformed equation cannot have a greater number of *positive* roots than it has *variations.*

Hence, the original equation cannot have a greater number of *negative* roots than it has *permanences.*

Note. In all applications of Descartes' Rule, the equation must contain a term independent of x; that is, no root must be equal to zero (Art. 282); for a zero root cannot be regarded as either positive or negative.

495. It follows from the last part of Art. 494 that in any equation, whether complete or incomplete, the number of negative roots cannot exceed the number of variations in the equation which is formed from the given equation by changing the signs of the alternate terms commencing with the second.

496. In any *complete* equation, the sum of the number of permanences and variations is equal to the number of terms less one, or to the degree of the equation.

That is, the sum of the number of permanences and variations is equal to the number of roots (Art. 473).

Hence, if the roots of a complete equation are all real, the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of permanences.

An equation whose terms are all positive can have no positive root.

A complete equation whose terms are alternately positive and negative can have no negative root.

497. 1. Determine the nature of the roots of

$$x^3 + 2x + 5 = 0.$$

There is no variation, and consequently no positive root.

Changing the signs of the alternate terms commencing with the second, we have $x^3 + 2x - 5 = 0$. (See Note, Art. 485.)

In this there are two variations, and hence the given equation cannot have more than two negative roots (Art. 495).

But the equation has three roots (Art. 473), and it cannot have an odd number of imaginary roots (Art. 482).

Hence, it has one negative and two imaginary roots.

Note. If two or more successive terms of an equation are wanting, it follows by Descartes' Rule that the equation must have imaginary roots.

EXAMPLES.

If the roots of the following equations are all real, determine their signs:

2. $2x^3 - 3x^2 - 17x + 30 = 0.$

3. $3x^3 - 11x^2 - 19x - 5 = 0.$

4. $x^4 - 8x^3 + 17x^2 + 2x - 24 = 0.$

5. $x^4 - 58x^2 + 441 = 0.$

6. $4x^4 + 28x^3 + 39x^2 - 7x - 10 = 0.$

7. $x^5 - 41x^3 + 12x^2 + 292x + 240 = 0.$

8. $3x^5 - 2x^4 - 45x^3 + 92x - 48 = 0.$

Determine the nature of the roots of the following:

9. $8x^3 - 27 = 0.$

10. $x^3 - 8x^2 - 12 = 0.$

11. $x^4 + 3x^2 + 1 = 0.$

12. $x^5 + 1 = 0.$

13. $x^5 - 4x^2 + 5 = 0.$

14. $x^5 + 2x^4 + 3x^2 + 1 = 0.$

DERIVATIVES.

498. If we take the polynomial

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots,$$

multiply each term by the exponent of x in that term, and then diminish the exponent by 1, the result

$$nax^{n-1} + (n-1)bx^{n-2} + (n-2)cx^{n-3} + \cdots$$

is called the *first derivative* of the given polynomial.

The first derivative of the first derivative is called the *second derivative* of the given polynomial; the first derivative of the second derivative is called the *third derivative;* and so on.

1. Find the successive derivatives of $3x^3-9x^2-12x+2$.

First, $9x^2 - 18x - 12$.
Second, $18x - 18$.
Third, 18.
Fourth, 0.

Note. We shall hereafter speak of the *first derivative* of an expression as *the derivative* of the expression.

EXAMPLES.

Find the successive derivatives of:

2. $2x^2 + x + 1$.
3. $x^3 - 5x^2 + 4x$.
4. $3x^4 + 8x^3 - 12x^2$.
5. $x^4 - x^3 - 3x^2 + 7$.
6. $2x^5 + 9x^3 - 21x$.
7. $5x^5 - 4x^4 + 3x^3 - 2$.

MULTIPLE ROOTS.

499. If an equation has two or more roots equal to a (Art. 473, Note), a is said to be a **Multiple Root** of the equation.

In the above case, a is called a *double root, triple root, quadruple root,* etc., according as the equation has two roots, three roots, four roots, etc., equal to a.

500. Let the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0 \tag{1}$$

be $a, b, c, d, \cdots$.

Then, by Art. 476, we have

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots = (x-a)(x-b)(x-c)\cdots.$$

Putting $x+h$ in place of x, we obtain

$$(x+h)^n + p_1(x+h)^{n-1} + p_2(x+h)^{n-2} + \cdots$$
$$= (h+\overline{x-a})(h+\overline{x-b})(h+\overline{x-c})\cdots. \tag{2}$$

Expanding the powers of $x+h$ by the Binomial Theorem, the coefficient of h in the first member of (2) is

$$nx^{n-1} + p_1(n-1)x^{n-2} + p_2(n-2)x^{n-3} + \cdots; \tag{3}$$

which, we observe, is the first derivative of the first member of (1).

Again, it is evident from Art. 477 that the coefficient of h in the second member of (2) is

$$\begin{aligned} &(x-b)(x-c)(x-d)\cdots \text{to } n-1 \text{ factors} \\ &+(x-a)(x-c)(x-d)\cdots \text{to } n-1 \text{ factors} \\ &+(x-a)(x-b)(x-d)\cdots \text{to } n-1 \text{ factors} + \cdots. \end{aligned} \tag{4}$$

Since equation (2) is true for every value of h, by Art. 377 these coefficients of h in the two members are equal.

Now if $b = a$, that is, if equation (1) has *two* roots equal to a, every term of (4) will be divisible by $x-a$, and therefore the expression (3) will be divisible by $x-a$.

Hence, the equation formed by equating (3) to zero will have *one* root equal to a (Art. 470).

In like manner, if $c = b = a$, that is, if (1) has *three* roots equal to a, the equation formed by equating (3) to zero will have *two* roots equal to a; and so on.

Hence, *if any equation of the form* (1) *has m roots equal to a, the equation formed by equating to zero the derivative of its first member will have $m-1$ roots equal to a.*

501. It follows from Art. 500 that, to determine the existence of multiple roots in an equation of the form

$$p_0x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0,$$

we proceed as follows:

Find the H. C. F. of the first member and its derivative.

If there is no H. C. F., there can be no multiple roots.

If there is a H. C. F., by equating it to zero and solving the resulting equation, the required roots may be obtained.

It is to be observed that the number of times that each root occurs in the given equation exceeds by one the number of times that it occurs in the equation formed by equating the H. C. F. to zero.

1. Find all the roots of

$$x^5 + x^4 - 9x^3 - 5x^2 + 16x + 12 = 0. \qquad (1)$$

The derivative of the first member is

$$5x^4 + 4x^3 - 27x^2 - 10x + 16.$$

The H. C. F. of this and the first member of (1) is $x^2 - x - 2$.

Solving the equation $x^2 - x - 2 = 0$, we have $x = 2$ or -1.

Hence, the multiple roots of (1) are 2, 2, -1, and -1.

Adding the sum of 2, 2, -1, and -1 to 1, and changing the sign of the result, the remaining root is -3 (Art. 479).

Therefore, the roots of (1) are 2, 2, -1, -1, and -3.

EXAMPLES.

Find all the roots of the following:

2. $x^3 + 3x^2 - 24x + 28 = 0.$
3. $x^3 - 4x^2 - 11x - 6 = 0.$
4. $8x^3 + 4x^2 - 66x + 63 = 0.$
5. $x^4 + 6x^3 + x^2 - 24x + 16 = 0.$
6. $x^4 + 7x^3 + 9x^2 - 27x - 54 = 0.$
7. $x^5 - 7x^3 + 2x^2 + 12x - 8 = 0.$
8. $x^4 - 6x^3 - 28x^2 + 120x + 288 = 0.$

502. The equation $x^n - a = 0$ can have no multiple roots; for the derivative of $x^n - a$ is nx^{n-1}, and $x^n - a$ and nx^{n-1} have no common factor except unity.

Hence, the n roots of $x^n = a$ are all different.

It follows from this that every expression has two different square roots, three different cube roots, and, in general, n different nth roots.

LOCATION OF ROOTS.

503. *If two real numbers, a and b, not roots of the equation*

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0, \tag{1}$$

when substituted for x in the first member, give results of opposite sign, an odd number of roots of the equation lie between a and b.

Let a be *algebraically* greater than b.

Let $d, \cdots, g$ be the real roots of (1) lying betwen a and b, and $h, \cdots, k$ the remaining real roots.

Let $x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n$ be denoted by X.

Then by Art. 476,

$$X = (x-d) \cdots (x-g) \cdot (x-h) \cdots (x-k) \cdot Y; \tag{2}$$

where Y denotes the product of the factors corresponding to the imaginary roots, if any, of (1).

Substituting a, and then b, for x in (2), the second member becomes

$$(a-d) \cdots (a-g) \cdot (a-h) \cdots (a-k) \cdot Y', \tag{3}$$

and. $$(b-d) \cdots (b-g) \cdot (b-h) \cdots (b-k) \cdot Y''; \tag{4}$$

where Y' and Y'' denote the values of Y when x is put equal to a and b, respectively.

Since a is greater than b, each of the quantities $d, \cdots, g$ is less than a and greater than b; whence, each of the factors $a-d, \cdots, a-g$ is $+$, and each of the factors $b-d, \cdots, b-g$ is $-$.

Again, since none of the quantities $h, \cdots, k$ lie between a and b, the expression $(a-h)\cdots(a-k)$ has the same sign as $(b-h)\cdots(b-k)$.

Also, Y' and Y'' are positive; for the product of the factors corresponding to a pair of conjugate imaginary roots of (1) is positive for every real value of x (Art. 483).

But by hypothesis, the expressions (3) and (4) are of opposite sign.

Therefore, the number of factors $b-d, \cdots, b-g$ must be *odd;* that is, an odd number of roots of (1) lie between a and b.

Note. If the numbers substituted differ by unity, it is evident that the integral part of at least one root is known.

1. Locate the roots of $x^3+x^2-6x-7=0$.

By Descartes' Rule (Art. 494), the equation cannot have more than one positive, nor more than two negative roots.

The values of the first member for the values 0, 1, 2, 3, -1, -2, and -3 of x are as follows:

$x=0$; -7. $x=2$; -7. $x=-1$; -1. $x=-3$; -7.

$x=1$; -11. $x=3$; 11. $x=-2$; 1.

Since the sign of the first member is $-$ when $x=2$, and $+$ when $x=3$, an odd number of roots lie between 2 and 3.

But the equation cannot have more than one positive root, and therefore *one* root lies between 2 and 3.

In like manner, another root lies between -1 and -2, and a third between -2 and -3.

The integral parts of the roots are 2, -1, and -2.

Note. In locating roots by the above method, first make trial of the numbers 0, 1, 2, etc., continuing the process until the number of positive roots determined is the same as has been previously indicated by Descartes' Rule.

Thus, in Ex. 1, we know by Descartes' Rule that the equation cannot have more than one positive root; and when one has been found to lie between 2 and 3, there is no need of trying 4, or any greater positive number.

EXAMPLES.

Locate the roots of the following equations:

2. $x^3 - 5x^2 + 3 = 0.$
3. $x^3 - 5x^2 + 2x + 6 = 0.$
4. $x^3 + 2x^2 - x - 1 = 0.$
5. $x^4 - 8x^2 + 15 = 0.$
6. $x^3 + 8x^2 - 9x - 12 = 0.$
7. $x^4 - 15x^2 + 3x + 14 = 0.$
8. $x^4 + 6x^3 - 42x - 44 = 0.$
9. $x^4 - 5x^3 + x^2 + 13x - 7 = 0.$

10. Prove that the equation $x^3 \quad x^2 + 2x - 1 = 0$ has at least one root between 0 and 1.

11. Prove that the equation $x^3 + 3x - 5 = 0$ has one root between 1 and 2.

12. Prove that the equation $x^4 - 2x^3 - 3x^2 + x - 2 = 0$ has one root between -1 and -2, and at least one between 2 and 3.

13. Prove that the equation $x^4 - 3x^3 + 6x^2 + x - 1 = 0$ has one root between 0 and -1, and at least one between 0 and 1.

504. The method of Art. 503 is not sufficient to deal with every problem in location of roots.

Let it be required, for example, to locate the roots of

$$x^3 + 3x^2 + 2x + 1 = 0.$$

By Art. 484, the equation has at least one real root.

By Descartes' Rule, it has no positive root.

Putting x equal to 0, -1, -2, and -3, the corresponding values of the first member are 1, 1, 1, and -5.

Then an odd number of roots lie between -2 and -3.

The equation can be written $x^2(x + 3) + 2x + 1 = 0$; from which it is evident that if x is algebraically less than -3, the first member is negative.

Then no root can be algebraically less than -3.

Thus either one or three roots lie between -2 and -3; but the methods previously given are insufficient to determine which.

Sturm's Theorem (Art. 505) affords a method for determining completely the number and situation of the real roots of an equation.

It is more difficult of application than the method of Art. 503; and should be used only in cases which the latter cannot resolve.

505. Sturm's Theorem.

Let $$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0 \qquad (1)$$
be an equation from which the multiple roots have been removed (Art. 501).

Let $x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n$ be denoted by X, and let X_1 denote the first derivative of X (Art. 498).

Dividing X by X_1, we shall obtain a quotient Q_1, with a remainder of a degree lower than that of X.

Denote this remainder, *with the sign of each of its terms changed,* by X_2, and divide X_1 by X_2, and so on; the operation being precisely the same as that of finding the H. C. F. of X and X_1, except that the signs of the terms of each remainder are to be changed, while no other changes of sign are permissible.

Since, by hypothesis, $X = 0$ has no multiple roots, X and X_1 have no common divisor except 1 (Art. 501); and we shall finally obtain a remainder X_n independent of x.

The expressions X, X_1, X_2, $\cdots$, X_n are called *Sturm's Functions.*

The successive operations are represented as follows:

$$X = Q_1X_1 - X_2, \qquad (2)$$
$$X_1 = Q_2X_2 - X_3, \qquad (3)$$
$$X_2 = Q_3X_3 - X_4, \qquad (4)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$X_{n-2} = Q_{n-1}X_{n-1} - X_n.$$

We may now enunciate Sturm's Theorem:

Let two real numbers, a and b, be substituted in place of x in Sturm's Functions, and the signs noted.

The difference between the number of variations of sign (Art. 493) *in the first case and that in the second is equal to the number of real roots of* $X=0$ *lying between* a *and* b.

The proof of the theorem depends upon the following principles:

I. *Two consecutive functions cannot both become* 0 *for the same value of* x.

For if, for any value of x, $X_1=0$ and $X_2=0$, then by (3), $X_3=0$; and since $X_2=0$ and $X_3=0$, by (4), $X_4=0$; continuing in this way, we shall finally have $X_n=0$.

But by hypothesis, X_n is independent of x, and consequently cannot become 0 for any value of x.

Hence, no two consecutive functions can become 0 for the same value of x.

II. *If any function, except* X *and* X_n, *becomes* 0 *for any value of* x, *the adjacent functions have opposite signs for this value of* x.

For if, for any value of x, $X_2=0$, then by (3) we must have $X_1=-X_3$ for this value of x.

Therefore, X_1 and X_3 must have opposite signs for this value of x; for, by I., neither of them can equal zero.

III. Let c be a root of the equation $X_r=0$, where X_r is any function except X and X_n.

Then, by II., X_{r-1} and X_{r+1} have opposite signs when $x=c$.

Let h be a positive quantity, so taken that no root of $X_{r-1}=0$ or $X_{r+1}=0$ lies between $c-h$ and $c+h$.

Then, as x changes from $c-h$ to $c+h$, no change of sign takes place in X_{r-1} or X_{r+1}, while X_r reduces to zero, and changes sign.

Therefore, for values of x between $c-h$ and c, the three functions X_{r-1}, X_r, and X_{r+1} present one permanence and one variation; while for values of x between c and $c+h$, they present one variation and one permanence.

Hence, as x increases from $c-h$ to $c+h$, no change occurs in the *number* of variations in the functions X_{r-1}, X_r, and X_{r+1}; that is, no change occurs in the number of variations as x increases through a root of $X_r = 0$.

IV. Let c be a root of the equation $X=0$; and let h be a positive quantity so taken that no root of $X_1 = 0$ lies between $c-h$ and $c+h$.

Then as x increases from $c-h$ to $c+h$, no change of sign takes place in X_1, while X reduces to zero, and changes sign.

Now if we put $x = c-h$ in (1), the first member becomes

$$(c-h)^n + p_1(c-h)^{n-1} + \cdots + p_{n-1}(c-h) + p_n.$$

Expanding the powers of $c-h$ by the Binomial Theorem, and collecting the terms involving like powers of h, we have

$$\begin{aligned} c^n + p_1c^{n-1} + \cdots + p_{n-1}c + p_n \\ - h[nc^{n-1} + (n-1)p_1c^{n-2} + \cdots + p_{n-1}] \\ + \text{terms involving } h^2, h^3, \cdots, h^n. \end{aligned} \tag{5}$$

But since c is a root of $X=0$, we have by (1),

$$c^n + p_1c^{n-1} + \cdots + p_{n-1}c + p_n = 0.$$

Also, it is evident that the coefficient of $-h$ is the value of X_1 when c is substituted in place of x; let this be denoted by A; then (5) reduces to

$$-hA + \text{terms involving } h^2, h^3, \cdots, h^n. \tag{6}$$

In like manner, the value of X when x is put equal to $c+h$, is

$$+hA + \text{terms involving } h^2, h^3, \cdots, h^n. \tag{7}$$

Now if h be taken sufficiently small, the signs of the expressions (6) and (7) will be the same as the signs of their first terms, $-hA$ and $+hA$, respectively.

Hence, if h be taken sufficiently small, the sign of (6) will be contrary to the sign of A, and the sign of (7) will be the same as the sign of A.

Therefore, for values of x between $c-h$ and c, the functions X and X_1 present a variation, and for values of x between c and $c+h$ they present a permanence.

Hence, a variation is lost as x increases through a root of $X=0$.

We may now prove Sturm's Theorem; for as x increases from b to a, supposing a *algebraically* greater than b, a variation is lost each time that x passes through a root of $X=0$, and only then; for when x passes through a root of $X_r=0$, where X_r is any function except X and X_n, no change occurs in the number of variations.

Hence, the number of variations lost as x increases from b to a is equal to the number of real roots of $X=0$ included between a and b.

506. It is customary, in applying Sturm's Theorem, to speak of the substitution of an indefinitely great number for x, in an expression, as *substituting* ∞ *for* x.

The substitution of $+\infty$ and $-\infty$ for x in Sturm's Functions determines the number of real roots of $X=0$.

The substitution of $+\infty$ and 0 for x determines the number of positive real roots, and the substitution of $-\infty$ and 0 for x determines the number of negative real roots.

507. If a sufficiently great number be substituted in place of x in the expression

$$X=p_0x^n+p_1x^{n-1}+\cdots+p_{n-1}x+p_n,$$

the sign of the result will be the same as the sign of its first term, p_0x^n.

It follows from the above that:

If $+\infty$ *be substituted in place of* x *in* X, *the sign of the result will be the same as the sign of its first term.*

If $-\infty$ *be substituted in place of* x *in* X, *the sign of the result will be the same as, or contrary to, the sign of its first term, according as the degree of* X *is even or odd.*

508. In the process of finding X_2, X_3, etc., any *positive* numerical factor may be omitted or introduced at pleasure; for the *sign* of the result is not affected thereby.

In this way fractions may be avoided.

509. Since Sturm's Theorem determines the number of real roots of an equation, the number of imaginary roots also becomes known (Art. 473).

510. 1. Determine the number and situation of the real roots of

$$x^3 - 6x^2 + 5x + 13 = 0.$$

Here, $X = x^3 - 6x^2 + 5x + 13$, and $X_1 = 3x^2 - 12x + 5$.

Multiplying X by 3 in order to make its first term divisible by $3x^2$, we have

$$\begin{array}{r|l}
3x^2 - 12x + 5\,) & 3x^3 - 18x^2 + 15x + 39\,(x - 2 \\
 & \underline{3x^3 - 12x^2 + 5x \phantom{{}+39}} \\
 & - 6x^2 + 10x + 39 \\
 & \underline{- 6x^2 + 24x - 10} \\
 & 7\,)\, \underline{-14x + 49} \\
 & - 2x + 7
\end{array}$$

$$\therefore\ X_2 = 2x - 7.$$

$$\begin{array}{r|l}
 & 3x^2 - 12x + 5 \\
 & \underline{2} \\
2x - 7\,) & 6x^2 - 24x + 10\,(3x \\
 & \underline{6x^2 - 21x} \\
 & - 3x + 10 \\
 & \underline{2} \\
 & - 6x + 20\,(-3 \\
 & \underline{- 6x + 21} \\
 & - 1
\end{array}$$

$$\therefore\ X_3 = 1.$$

Substituting $-\infty$ for x in X, X_1, X_2, and X_3, the signs are $-$, $+$, $-$, and $+$, respectively (Art. 507); substituting 0 for x, the signs are $+$, $+$, $-$, and $+$, respectively; and substituting $+\infty$ for x, the signs are all $+$.

Hence, the roots of the equation are all real; two of them are positive, and the other negative.

We now substitute various numbers to determine the situation of the roots:

	X	X_1	X_2	X_3	
$x=-\infty$,	$-$	$+$	$-$	$+$	3 variations.
$x=-2$,	$-$	$+$	$-$	$+$	3 variations.
$x=-1$,	$+$	$+$	$-$	$+$	2 variations.
$x=0$,	$+$	$+$	$-$	$+$	2 variations.
$x=1$,	$+$	$-$	$-$	$+$	2 variations.
$x=2$,	$+$	$-$	$-$	$+$	2 variations.
$x=3$,	$+$	$-$	$-$	$+$	2 variations.
$x=4$,	$+$	$+$	$+$	$+$	no variation.
$x=\infty$,	$+$	$+$	$+$	$+$	no variation.

We then know that the equation has one root between -1 and -2, and two roots between 3 and 4.

Note. In substituting the numbers, it is best to work from 0 in either direction, stopping when the number of variations is the same as has been previously found for $+\infty$ or $-\infty$, as the case may be.

2. Determine the number and situation of the real roots of

$$X = 4x^3 - 2x - 5 = 0.$$

Here, $X_1 = 12x^2 - 2$; or, $6x^2 - 1$, omitting the factor 2

$$\begin{array}{r}
4x^3 - 2x - 5 \\
3 \\
\hline
6x^2 - 1)\,12x^3 - 6x - 15\,(2x \\
12x^3 - 2x \phantom{{}-15\,(2x} \\
\hline
-4x - 15
\end{array}$$

$$\therefore\ X_2 = 4x + 15.$$

$$
\begin{array}{r|l}
 & 6x^2 - 1 \\
 & 2 \\
\hline
4x+15) & 12x^2 - 2\,(3x \\
 & 12x^2 + 45x \\
\hline
 & -45x - 2 \\
 & 4 \\
\hline
 & -180x - 8\,(-45 \\
 & -180x - 675 \\
\hline
 & 667
\end{array}
$$

$$\therefore \; X_3 = -667.$$

	X	X_1	X_2	X_3	
$x = -\infty$,	$-$	$+$	$-$	$-$	2 variations.
$x = 0$,	$-$	$-$	$+$	$-$	2 variations.
$x = 1$,	$-$	$+$	$+$	$-$	2 variations.
$x = 2$,	$+$	$+$	$+$	$-$	1 variation.
$x = \infty$,	$+$	$+$	$+$	$-$	1 variation.

Therefore, the equation has a real root between 1 and 2, and two imaginary roots.

EXAMPLES.

Determine the number and situation of the real roots of:

3. $x^3-4x^2-4x+12=0$.
4. $x^3+5x+2=0$.
5. $x^3-x^2-2x+1=0$.
6. $x^3+3x^2-9x-4=0$.
7. $x^3-4x^2-10x+41=0$.
8. $x^4-12x^2+12x-3=0$.
9. $2x^4-3x^2+3x-1=0$.
10. $x^4+2x^3-6x^2-8x+9=0$.

XLI. SOLUTION OF HIGHER EQUATIONS.

511. Synthetic Division.

The operation of division, in the examples of the present chapter, may be conveniently performed by a process known as *Synthetic Division.*

In finding the quotient of two expressions which are arranged according to the same order of powers of some common letter, the operation may be abridged by writing only the *numerical coefficients* and *signs* of the terms.

Thus, let it be required to divide $x^3 - 12x^2 + 29x - 21$ by $x - 3$.

$$\begin{array}{rrrr|l} 1 & -12 & +29 & -21 & 1-3 \\ \cline{5-5} 1 & -3 & & & 1-9+2,\ \text{Quotient.} \\ \cline{1-2} & -9 & & & \\ & -9 & +27 & & \\ \cline{2-3} & & +2 & & \\ & & +2 & -6 & \\ \cline{3-4} & & & -15, & \text{Remainder.} \end{array}$$

We may omit the first term of each partial product, for it is merely a repetition of the term immediately above.

Also, the second term of each partial product may be *added* to the corresponding term of the dividend, provided we change the sign of the second term of the divisor before multiplying.

The work now stands:

$$\begin{array}{rrrr|l} 1 & -12 & +29 & -21 & 1+3 \\ \cline{5-5} & +3 & & & 1-9+2 \\ \cline{2-2} & -9 & & & \\ & & -27 & & \\ \cline{3-3} & & +2 & & \\ & & & +6 & \\ \cline{4-4} & & & -15 & \end{array}$$

The first term of the divisor being unity in all applications of the method, it may be omitted; and the first terms of the successive dividends constitute the quotient.

Raising the oblique columns, the operation will stand as follows:

$$\begin{array}{llrrrl} \text{Dividend,} & 1 & -12 & +29 & -21 & \underline{|+3} \\ \text{Partial products,} & & \underline{+\ 3} & \underline{-27} & \underline{+\ 6} & \\ \text{Quotient,} & 1 & -\ 9 & +\ 2, & -15 & \text{Remainder.} \end{array}$$

The complete result is obtained as follows:

Multiplying the first term of the dividend by 3, and adding the result to the second term of the dividend, gives the second term of the quotient.

Multiplying the latter by 3, and adding the result to the third term of the dividend, gives the last term of the quotient.

Multiplying the latter by 3, and adding the result to the last term of the dividend, gives the remainder.

Therefore, the quotient is $x^2 - 9x + 2$, and the remainder -15.

Note. If the term involving any power is wanting, it must be supplied with the coefficient 0 before applying the rule.

COMMENSURABLE ROOTS.

512. A *commensurable* root is one which can be exactly expressed as an integer or fraction, without using irrational quantities.

513. We know, by Art. 480, that an equation of the nth degree in its general form (Art. 467), whose coefficients are integral, cannot have as a root a rational fraction in its lowest terms.

Therefore, to find all the commensurable roots of such an equation, we have only to find all its integral roots.

Again, by Art. 478, the last term of an equation of the above form is divisible by every integral root.

Hence, to find all the commensurable roots, we have only to *ascertain by trial which integral divisors of the last term are roots of the equation.*

The trial may be made in two ways:

I. By actual substitution of the supposed root.

II. By dividing the first member of the equation by the unknown quantity minus the supposed root (Art. 469).

In this case, the operation may be conveniently performed by Synthetic Division (Art. 511).

In the case of small numbers, such as ± 1, the first method may be the more convenient.

The second method has the advantage that, when a root has been found, the process gives at once the depressed equation (Art. 475) for obtaining the remaining roots.

Descartes' Rule of Signs (Art. 494) may be advantageously employed to shorten the process.

Any multiple root should be removed (Art. 501) before applying either method.

514. 1. Find all the roots of $x^4 - 15x^2 + 10x + 24 = 0$.

By Descartes' Rule, the equation cannot have more than two positive roots.

Changing the signs of the alternate terms commencing with the second, we have $x^4 - 15x^2 - 10x + 24 = 0$.

Hence, the given equation cannot have more than two negative roots (Art. 495).

The integral divisors of 24 are ± 1, ± 2, ± 3, ± 4, ± 6, ± 8, ± 12, and ± 24.

By actual substitution, we find that 1 is not, and that -1 is, a root of the equation.

Dividing the first member by $x - 2$, $x - 3$, etc., by the method explained in Art. 511, we have

$$\begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{2} \\ & 2 & 4 & -22 & -24 & \\ \hline & 2 & -11 & -12, & 0 & \text{Rem.} \end{array} \qquad \begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{3} \\ & 3 & 9 & -18 & -24 & \\ \hline & 3 & -6 & -8, & 0 & \text{Rem.} \end{array}$$

The work shows that 2 and 3 are roots of the given equation; and since the equation cannot have more than two positive roots, these are the only positive roots, and there is no need of trying the numbers 4, 6, 8, 12, and 24.

The remaining root may be found by dividing 24 by the product of -1, 2, and 3 (Art. 479), or by the same process as before.

Dividing the first member by $x+2$, $x+3$, etc., we have

$$\begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{-2} \\ & -2 & 4 & 22 & -64 & \\ \hline & -2 & -11 & 32, & -40 & \text{Rem.} \end{array} \qquad \begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{-4} \\ & -4 & 16 & -4 & -24 & \\ \hline & -4 & 1 & 6, & 0 & \text{Rem.} \end{array}$$

$$\begin{array}{rrrrr|l} 1 & +0 & -15 & +10 & +24 & \underline{-3} \\ & -3 & 9 & 18 & -84 & \\ \hline & -3 & -6 & 28, & -60 & \text{Rem.} \end{array}$$

The work shows that the remaining root is -4.

Thus, the four roots of the equation are -1, 2, 3, and -4.

By Art. 487, an equation of the nth degree in its general form with fractional coefficients may be transformed into another whose coefficients are integral, that of the first term being unity.

The commensurable roots of the transformed equation may then be found as in the preceding example.

2. Find all the roots of $4x^3 - 12x^2 + 27x - 19 = 0$.

Dividing each term by the coefficient of x^3, we have

$$x^3 - 3x^2 + \frac{27x}{4} - \frac{19}{4} = 0.$$

Proceeding as in Art. 487, it is evident by inspection that the multiplier 2 will remove the fractional coefficients.

The transformed equation is

$$x^3 - 2\cdot 3x^2 + 2^2\cdot\frac{27x}{4} - 2^3\cdot\frac{19}{4} = 0,$$

or,
$$x^3 - 6x^2 + 27x - 38 = 0. \qquad (1)$$

The roots of this equation are those of the given equation multiplied by 2.

By Descartes' Rule, equation (1) has no negative root.

The positive integral divisors of 38 are 1, 2, 19, and 38.

Dividing the first member by $x-1$, $x-2$, etc., we have

$$\begin{array}{rrrrl} 1 & -6 & +27 & -38 & \underline{|1} \\ & \underline{1} & \underline{-5} & \underline{22} & \\ & -5 & 22, & -16 & \text{Rem.} \end{array} \qquad \begin{array}{rrrrl} 1 & -6 & +27 & -38 & \underline{|2} \\ & \underline{2} & \underline{-8} & \underline{38} & \\ & -4 & 19, & 0 & \text{Rem.} \end{array}$$

The work shows that 2 is a root of (1).

The other two roots may now be found by depressing the equation; it is evident from the right-hand operation above that the depressed equation is $x^2-4x+19=0$.

Solving this by the rules for quadratics, we have

$$x=2\pm\sqrt{4-19}.$$

Hence, the three roots of (1) are 2 and $2\pm\sqrt{-15}$.

Dividing by 2, the roots of the given equation are

$$1 \text{ and } 1\pm\tfrac{1}{2}\sqrt{-15}.$$

EXAMPLES.

Find all the commensurable roots of each of the following, and the remaining roots when possible by methods already given:

3. $x^3-8x^2+19x-12=0$.
4. $x^3-31x-30=0$.
5. $x^3+5x^2-6x-24=0$.
6. $2x^3+x^2-23x+20=0$.
7. $x^3-7x^2-14x+48=0$.
8. $3x^3+2x^2-3x-2=0$.
9. $x^4+2x^3-7x^2-8x+12=0$.
10. $x^4-x^3-7x^2+x+6=0$.
11. $x^4+6x^3+x^2-24x-20=0$.
12. $4x^4-12x^3+3x^2+13x-6=0$.
13. $x^4+11x^3+41x^2+61x+30=0$.

14. $x^4 + x^3 - 31x^2 + 71x - 42 = 0.$

15. $4x^4 - 31x^2 + 21x + 18 = 0.$

16. $x^4 - 11x^3 + 35x^2 - 13x - 60 = 0.$

17. $9x^4 - 16x^2 - 3x + 4 = 0.$

18. $x^4 - 7x^3 + 15x^2 - x - 24 = 0.$

RECIPROCAL OR RECURRING EQUATIONS.

515. A **Reciprocal Equation** is one such that if any quantity is a root of the equation, its reciprocal is also a root.

516. It follows from Art. 515 that, if $\frac{1}{x}$ be substituted for x in a reciprocal equation, the transformed equation will have the same roots as the given equation.

517. Let

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-2}x^2 + p_{n-1}x + p_n = 0 \quad (1)$$

be a reciprocal equation.

Substituting $\frac{1}{x}$ for x, the equation becomes

$$\frac{p_0}{x^n} + \frac{p_1}{x^{n-1}} + \frac{p_2}{x^{n-2}} + \cdots + \frac{p_{n-2}}{x^2} + \frac{p_{n-1}}{x} + p_n = 0;$$

or, by clearing of fractions, and reversing the order of the terms,

$$p_nx^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \cdots + p_2x^2 + p_1x + p_0 = 0. \quad (2)$$

By Art. 516, equation (2) has the same roots as (1).

Hence, the following relations must hold between the coefficients of (1) and (2):

$$p_0 = \pm p_n, \ p_1 = \pm p_{n-1}, \ p_2 = \pm p_{n-2}, \text{ etc.};$$

or, in general, $p_r = \pm p_{n-r};$

all the upper signs, or all the lower signs, being taken together.

We may then have four varieties of reciprocal equations:

1. Degree odd, and coefficients of terms equally distant from the extremes of the first member equal in absolute value and of *like* sign; as, $x^3 - 2x^2 - 2x + 1 = 0$.

2. Degree odd, and coefficients of terms equally distant from the extremes of the first member equal in absolute value and of *unlike* sign; as, $3x^5 + 2x^4 - x^3 + x^2 - 2x - 3 = 0$.

3. Degree even, and coefficients of terms equally distant from the extremes of the first member equal in absolute value and of *like* sign; as, $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$.

4. Degree even, and coefficients of terms equally distant from the extremes of the first member equal in absolute value and of *unlike* sign, and middle term wanting; as, $2x^6 + 3x^5 - 7x^4 + 7x^2 - 3x - 2 = 0$.

On account of the properties stated above, reciprocal equations are also called *Recurring Equations.*

518. Every reciprocal equation of the first variety may be written in the form

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_2x^2 + p_1x + p_0 = 0,$$

or, $$p_0(x^n+1) + p_1x(x^{n-2}+1) + p_2x^2(x^{n-4}+1) + \cdots = 0; \quad (1)$$

since the number of terms is even.

By Art. 100, since n is odd, each of the expressions $x^n + 1$, $x^{n-2} + 1$, etc., is divisible by $x + 1$.

Whence, -1 is a root of the equation (Art. 470).

Dividing the first member of (1) by $x + 1$, the depressed equation is

$$p_0(x^{n-1} - x^{n-2} + x^{n-3} - \cdots + x^2 - x + 1)$$
$$+ p_1x(x^{n-3} - x^{n-4} + x^{n-5} - \cdots + x^2 - x + 1)$$
$$+ p_2x^2(x^{n-5} - x^{n-6} + x^{n-7} - \cdots + x^2 - x + 1) + \cdots = 0,$$

or, $$p_0x^{n-1} + (p_1 - p_0)x^{n-2} + (p_2 - p_1 + p_0)x^{n-3} + \cdots$$
$$+ (p_2 - p_1 + p_0)x^2 + (p_1 - p_0)x + p_0 = 0;$$

which is a reciprocal equation of the third variety.

519. Every reciprocal equation of the second variety may be written in the form

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots - (\cdots + p_2x^2 + p_1x + p_0) = 0,$$

or, $p_0(x^n - 1) + p_1x(x^{n-2} - 1) + p_2x^2(x^{n-4} - 1) + \cdots = 0.$ (1)

Since each of the expressions $x^n - 1$, $x^{n-2} - 1$, etc., is divisible by $x - 1$ (Art. 100), $+1$ is a root of the equation.

Dividing the first member of (1) by $x - 1$, the depressed equation is

$$p_0x^{n-1} + (p_1 + p_0)x^{n-2} + (p_2 + p_1 + p_0)x^{n-3} + \cdots$$
$$+ (p_2 + p_1 + p_0)x^2 + (p_1 + p_0)x + p_0 = 0;$$

which is a reciprocal equation of the third variety.

520. Every reciprocal equation of the fourth variety may be written in the form

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots - (\cdots + p_2x^2 + p_1x + p_0) = 0,$$

or, $\quad p_0(x^n - 1) + p_1x(x^{n-2} - 1) + p_2x^2(x^{n-4} - 1) + \cdots = 0;$ (1)

since the number of terms is even (Art. 517).

Since each of the expressions $x^n - 1$, $x^{n-2} - 1$, etc., is divisible by $x^2 - 1$, both 1 and -1 are roots of the equation.

Dividing the first member of (1) by $x^2 - 1$, the depressed equation is

$$p_0(x^{n-2} + x^{n-4} + \cdots + x^4 + x^2 + 1)$$
$$+ p_1x(x^{n-4} + x^{n-6} + \cdots + x^4 + x^2 + 1)$$
$$+ p_2x^2(x^{n-6} + x^{n-8} + \cdots + x^4 + x^2 + 1) + \cdots = 0,$$

or, $\quad p_0x^{n-2} + p_1x^{n-3} + (p_2 + p_0)x^{n-4} + \cdots$

$$+ (p_2 + p_0)x^2 + p_1x + p_0 = 0;$$

which is a reciprocal equation of the third variety.

521. *Every reciprocal equation of the third variety may be reduced to an equation of half its degree.*

Let the equation be

$$p_0x^{2m} + p_1x^{2m-1} + \cdots + p_mx^m + \cdots + p_1x + p_0 = 0.$$

Dividing each term by x^m, the equation may be written

$$p_0\left(x^m+\frac{1}{x^m}\right)+p_1\left(x^{m-1}+\frac{1}{x^{m-1}}\right)+\cdots$$
$$+p_{m-2}\left(x^2+\frac{1}{x^2}\right)+p_{m-1}\left(x+\frac{1}{x}\right)+p_m=0. \quad (1)$$

Put $\quad x+\dfrac{1}{x}=y.$

Then, $x^2+\dfrac{1}{x^2}=\left(x+\dfrac{1}{x}\right)^2-2=y^2-2;$

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)-\left(x+\frac{1}{x}\right)$$
$$=y(y^2-2)-y=y^3-3y;$$
$$x^4+\frac{1}{x^4}=\left(x+\frac{1}{x}\right)\left(x^3+\frac{1}{x^3}\right)-\left(x^2+\frac{1}{x^2}\right)$$
$$=y(y^3-3y)-(y^2-2)=y^4-4y^2+2; \text{ etc.}$$

The general law is expressed by

$$x^r+\frac{1}{x^r}=\left(x+\frac{1}{x}\right)\left(x^{r-1}+\frac{1}{x^{r-1}}\right)-\left(x^{r-2}+\frac{1}{x^{r-2}}\right);$$

which is an expression of the rth degree with respect to y.

Substituting these values in (1), the equation takes the form

$$q_0y^m+q_1y^{m-1}+q_2y^{m-2}+\cdots=0.$$

522. It follows from Arts. 518 to 521 that any reciprocal equation of the degree $2m+1$, and any reciprocal equation of the fourth variety of the degree $2m+2$, can always be reduced to an equation of the mth degree.

523. 1. Solve $2x^5-5x^4-13x^3+13x^2+5x-2=0.$

This being a reciprocal equation of the second variety, one root is 1 (Art. 519).

Dividing the first member by $x-1$, the depressed equation is

$$2x^4-3x^3-16x^2-3x+2=0;$$

a reciprocal equation of the third variety.

Dividing each term by x^2, the equation becomes

$$2\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)-16=0.$$

Putting $x+\frac{1}{x}=y$, and $x^2+\frac{1}{x^2}=y^2-2$ (Art. 521), we have

$$2(y^2-2)-3y-16=0.$$

Solving this equation, $y=4$ or $-\frac{5}{2}$.

Taking the first value, $x+\frac{1}{x}=4$, or $x^2-4x=-1$.

Whence, $x=2\pm\sqrt{3}$.

Taking the second value, $x+\frac{1}{x}=-\frac{5}{2}$, or $2x^2+5x=-2$.

Whence, $x=-2$ or $-\frac{1}{2}$.

Thus, the roots of the given equation are 1, -2, $-\frac{1}{2}$, and $2\pm\sqrt{3}$.

Note. That $2+\sqrt{3}$ and $2-\sqrt{3}$ are reciprocals may be shown by multiplying them together; thus, $(2+\sqrt{3})(2-\sqrt{3})=4-3=1$.

EXAMPLES.

Solve the following equations:

2. $6x^3-7x^2-7x+6=0.$

3. $x^3+5x^2-5x-1=0.$

4. $5x^4+26x^3-26x-5=0.$

5. $x^3-ax^2+ax-1=0.$

6. $45x^4-48x^3-250x^2-48x+45=0.$

7. $x^5-29x^3+29x^2-1=0.$

8. $x^5 + 7x^4 + x^3 + x^2 + 7x + 1 = 0.$

9. $x^6 - x^5 + x^4 - x^2 + x - 1 = 0.$

10. $24x^5 - 34x^4 - 67x^3 + 67x^2 + 34x - 24 = 0.$

11. $3x^5 + 16x^4 + 29x^3 + 29x^2 + 16x + 3 = 0.$

524. Binomial Equations.

A *Binomial Equation* is an equation of the form $x^n = a$.

Binomial equations are also reciprocal equations, and may, in certain cases, be solved by the method of Art. 523.

EXAMPLES.

Solve the following equations:

1. $x^5 = 1.$ 2. $x^5 = -1.$ 3. $x^5 = 32.$ (Put $x = 2y$.)

CUBIC EQUATIONS.

525. A **Cubic Equation** is an equation of the *third degree*, involving but one unknown quantity.

526. By Art. 492, the cubic equation

$$x^3 + p_1x^2 + p_2x + p_3 = 0,$$

where p_1 is not zero, may be transformed into another whose second term shall be wanting by substituting $y - \frac{p_1}{3}$ in place of x.

Therefore, every cubic equation can be reduced to the form

$$x^3 + ax + b = 0.$$

527. Cardan's Method for the Solution of Cubics.

Let it be required to solve the equation $x^3 + ax + b = 0$.

Putting $x = y + z$, the equation becomes

$$y^3 + 3yz(y + z) + z^3 + a(y + z) + b = 0,$$

or, $$y^3 + z^3 + (3yz + a)(y + z) + b = 0.$$

We may give such a value to z that $3yz + a$ shall be equal to zero.

Whence, $$z = -\frac{a}{3y}. \qquad (1)$$

Then, $$y^3 + z^3 + b = 0. \qquad (2)$$

Substituting the value of z from (1) in (2), we have

$$y^3 - \frac{a^3}{27y^3} + b = 0, \text{ or } y^6 + by^3 = \frac{a^3}{27}.$$

This is an equation in the quadratic form (Art. 269).

Solving by the rules for quadratics, we have

$$y^3 = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}. \qquad (3)$$

Then by (2), $$z^3 = -y^3 - b = -\frac{b}{2} \mp \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}. \qquad (4)$$

Since $x = y + z$, the values of x corresponding to the upper and lower signs in (3) and (4) will evidently be the same.

Therefore,

$$x = \sqrt[3]{\left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)} + \sqrt[3]{\left(-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)}. \qquad (5)$$

The other two roots may be found by depressing the given equation (Art. 475).

528. It follows from Art. 527 that

To solve a cubic equation of the form $x^3 + ax + b = 0$, *we substitute* $y - \dfrac{a}{3y}$ *for* x.

529. 1. Solve the equation $x^3 - 3x^2 - 6x - 20 = 0$.

We first transform the equation into another whose second term shall be wanting.

Putting $x = y - \dfrac{-3}{3} = y + 1$ (Art. 526), we have

$$y^3 + 3y^2 + 3y + 1 - 3y^2 - 6y - 3 - 6y - 6 - 20 = 0,$$

or, $$y^3 - 9y - 28 = 0. \qquad (1)$$

In this case, $a = -9$ and $b = -28$.
Substituting in (5), Art. 527, we have

$$y = \sqrt[3]{14 + \sqrt{196 - 27}} + \sqrt[3]{14 - \sqrt{196 - 27}}$$
$$= \sqrt[3]{27} + \sqrt[3]{1} = 3 + 1 = 4.$$

Whence, $$x = y + 1 = 5.$$

Dividing the first member of the given equation by $x - 5$, the depressed equation is

$$x^2 + 2x + 4 = 0.$$

Solving, $$x = -1 \pm \sqrt{-3}.$$

Thus, the roots of the given equation are 5 and $-1 \pm \sqrt{-3}$.

Note. Equation (1) may also be solved by putting y equal to $z - \frac{-9}{3z}$ or $z + \frac{3}{z}$ (Art. 528).

EXAMPLES.

Solve the following equations:

2. $x^3 + 15x + 124 = 0.$
3. $x^3 - 27x - 54 = 0.$
4. $x^3 + 105x - 218 = 0.$
5. $x^3 - 6x^2 - 33x - 70 = 0.$
6. $x^3 - 9x^2 + 63x + 73 = 0.$
7. $x^3 + x^2 - 33x + 63 = 0.$
8. $x^3 + 12x^2 + 57x + 74 = 0.$
9. $x^3 - 4x^2 - 11x - 6 = 0.$
10. $x^3 - 2x^2 + 3 = 0.$
11. $x^3 + x^2 - 7x - 52 = 0.$
12. Find one root of $x^3 - x + 1 = 0.$

530. If a is negative, and $\frac{a^3}{27}$ numerically greater than $\frac{b^2}{4}$, the expression $\sqrt{\frac{b^2}{4} + \frac{a^3}{27}}$ is imaginary.

In such a case, Cardan's method is of no practical value; for there is no method in Algebra for finding the cube root of a binomial surd.

In this case, which is called the *Irreducible Case*, Cardan's method is said to *fail.*

It is possible, in cases where Cardan's method fails, to find the roots by a method involving Trigonometry; but practically it is easier to find them by the method of Art. 513, or by Horner's method (Art. 532), according as the equation has or has not a commensurable root.

INCOMMENSURABLE ROOTS.

531. We will now show how to find the approximate numerical values of those roots of an equation which are not commensurable (Art. 512).

532. Horner's Method of Approximation.

Let it be required to find the approximate value of the root between 3 and 4 of the equation

$$x^3 - 3x^2 - 2x + 5 = 0.$$

We first diminish the roots of the given equation by 3, by the second method explained in Art. 490.

The operation is conveniently performed by Synthetic Division (Art. 511).

$$\begin{array}{lrrrl}
 & 1 & -3 & -2 & +5\;\underline{|3} \\
 & & \underline{3} & \underline{0} & \underline{-6} \\
\text{1st quotient,} & \overline{1} & 0 & -2, & -1 \text{ 1st Rem.} \\
 & & \underline{3} & \underline{9} & \\
\text{2d quotient,} & \overline{1} & 3, & 7 \text{ 2d Rem.} & \\
 & & \underline{3} & & \\
 & & 6 \text{ 3d Rem.} & &
\end{array}$$

The transformed equation is $y^3 + 6y^2 + 7y - 1 = 0.$ (1)

We know that equation (1) has a root between 0 and 1.

If, then, we neglect the terms involving y^3 and y^2, we may obtain an approximate value of y by solving the equation $7y - 1 = 0$; thus, approximately, $y = .1$, and $x = 3.1$.

We now diminish the roots of (1) by .1; thus,

$$\begin{array}{rrrrl}
1 & +6 & +7 & -1 & \lfloor .1 \\
 & .1 & .61 & .761 & \\
 & 6.1 & 7.61 & -\ .239 & \\
 & .1 & .62 & & \\
 & 6.2 & 8.23 & & \\
 & .1 & & & \\
 & 6.3 & & &
\end{array}$$

The transformed equation is $z^3 + 6.3\,z^2 + 8.23\,z - .239 = 0$.
Neglecting the z^3 and z^2 terms, we have, approximately,

$$z = \frac{.239}{8.23} = .02+.$$

Thus, the value of x to two places of decimals is 3.12.

The work is usually arranged in the following form, the coefficients of the successive transformed equations being denoted by (1), (2), (3), etc.:

$$\begin{array}{rrrrl}
1 & -3 & -2 & +5 & \lfloor 3.128 \\
 & 3 & 0 & -6 & \\
 & 0 & -2 & (1)\ \ -1 & \\
 & 3 & 9 & .761 & \\
 & 3 & (1)\ \ 7 & (2)\ \ -\ .239 & \\
 & 3 & .61 & .167128 & \\
 & (1)\ \ 6 & 7.61 & (3)\ \ -\ .071872 & \\
 & .1 & .62 & & \\
 & 6.1 & (2)\ \ 8.23 & & \\
 & .1 & .1264 & & \\
 & 6.2 & 8.3564 & & \\
 & .1 & .1268 & & \\
 & (2)\ \ 6.3 & (3)\ \ 8.4832 & & \\
 & .02 & & & \\
 & 6.32 & & & \\
 & .02 & & & \\
 & 6.34 & & & \\
 & .02 & & & \\
 & (3)\ \ 6.36 & & &
\end{array}$$

Dividing .071872 by 8.4832, we have .008 suggested as the fourth figure of the root.

Thus, the value of x to three places of decimals is 3.128.

The process may be continued until the value of the root has been found to any desired degree of precision.

We derive from the above example the following rule for finding an approximate value of a positive incommensurable root:

Find by Art. 503, *or by Sturm's Theorem* (Art. 505), *the integral part of the root.* (Compare Art. 504.)

Transform the given equation into another whose roots shall be less by this integral part.

Divide the absolute value of the last term of the transformed equation by the absolute value of the coefficient of the first power of the unknown quantity, and write the approximate value of the result as the next figure of the root.

Transform the last equation into another whose roots shall be less by the figure of the root last obtained, and divide as before for the next figure of the root; and so on.

Note. In practice, the work may be contracted by dropping such decimal figures from the right of each column as are not needed for the required degree of accuracy.

533. To find an approximate value of a *negative* incommensurable root, change the signs of the alternate terms of the equation commencing with the second (Art. 485), and find the corresponding positive incommensurable root of the transformed equation.

The result with its sign changed will be the required negative root.

534. In finding any particular root-figure by the method of Art. 532, we are liable, especially in the first part of the process, to get too great a result; the same thing occasionally happens when extracting square or cube roots of numbers.

Such an error may be discovered by observing the signs of the last two terms of the transformed equation; for since each root-figure obtained as in Art. 532 must be *positive*, the last two terms of the transformed equation must be of *unlike sign*.

If this is not the case, the last root-figure must be diminished until a result is obtained which satisfies this condition.

Let it be required, for example, to find the root between 0 and -1 of the equation $x^3 + 4x^2 - 9x - 5 = 0$.

Changing the signs of the alternate terms commencing with the second (Art. 533), we have to find the root between 0 and 1 of the equation $x^3 - 4x^2 - 9x + 5 = 0$.

Dividing 5 by 9, we have .5 suggested as the first root-figure; but in this case the last two terms of the second transformed equation are -12.25 and $-.375$.

This shows that .5 is too great.

We then try .4, and find that the last two terms of the second transformed equation are of unlike sign.

The work of finding the first three root-figures is shown below:

1	-4		-9		$+5$	$\lfloor .469$
	$.4$		-1.44		-4.176	
	-3.6		-10.44	(1)	$.824$	
	$.4$		-1.28		$-.713064$	
	-3.2	(1)	-11.72	(2)	$-.110936$	
	$.4$		$-.1644$			
(1)	-2.8		-11.8844			
	$.06$		$-.1608$			
	-2.74	(2)	-12.0452			
	$.06$					
	-2.68					
	$.06$					
(2)	-2.62					

Then the required root is $-.469$, to three decimal places.

In any case the root-figure to be taken is *the greatest number which will ensure that the last two terms of the next transformed equation shall be of opposite sign.*

535. If the coefficient of the first power of the unknown quantity in any transformed equation is zero, the next root-figure may be found by *dividing the last term by the coefficient of the square of the unknown quantity, and taking the square root of the absolute value of the result.*

For if the transformed equation is $y^3 + ay^2 + b = 0$, it is evident that, approximately, $ay^2 + b = 0$, or $y = \sqrt{-\frac{b}{a}}$.

We proceed in a similar manner if any number of consecutive terms immediately preceding the last term are zero.

536. Horner's method may be used to find any root of a number approximately; for to find the nth root of a is the same thing as to solve the equation $x^n - a = 0$.

537. If an equation has two or more roots which have the same integral part, the first decimal root-figure of each must be obtained by the method of Art. 503, or by Sturm's Theorem.

If two or more roots have the same integral part, and also the same first decimal root-figure, the second decimal root-figure of each must be obtained by the method of Art. 503, or by Sturm's Theorem; and so on.

Note. If all but one of the roots of an equation are known, the remaining root may be found by adding the sum of the known roots to the coefficient of the second term, and changing the sign of the result (Art. 479).

EXAMPLES.

538. 1. Find the roots between 1 and 2, and -1 and -2, of

$$x^3 - 3x^2 - 2x + 5 = 0.$$

2. Find the root between 5 and 6 of

$$x^3 + 2x^2 - 23x - 70 = 0.$$

3. Find the root between -2 and -3 of

$$x^3 - 3x^2 - 3x + 18 = 0.$$

4. Find the root between 0 and 1 of

$$x^3 + 6x^2 + 10x - 1 = 0.$$

5. Find the root between -5 and -6 of

$$x^3 - x^2 - 25x + 81 = 0.$$

6. Find the root between 3 and 4 of

$$x^4 - 10x^2 - 4x + 8 = 0.$$

7. Find the root between -2 and -3 of

$$x^4 + 6x^3 + 12x^2 - 11x - 41 = 0.$$

8. Find the root between 0 and 1 of

$$x^4 + 3x^3 - 3x^2 + 19x - 12 = 0.$$

Find the real roots of the following:

9. $x^3 - 2x^2 - x + 1 = 0.$ **12.** $x^4 - 12x + 7 = 0.$

10. $x^3 - 3x - 1 = 0.$ **13.** $x^4 - x^3 + x - 2 = 0.$

11. $x^3 + 3x^2 + 4x + 5 = 0.$ **14.** $x^3 - 3x^2 - 4x + 13 = 0.$

Find the approximate values of the following (Art. 536):

15. $\sqrt[3]{2}.$ **16.** $\sqrt[3]{17}.$ **17.** $\sqrt[4]{5}.$

539. We may now give general directions for finding the real roots of any equation in the form

$$x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0,$$

with integral numerical coefficients:

1. Determine by Descartes' Rule (Art. 494) limits to the number of positive and negative roots.

2. Find all the commensurable roots, if any, as explained in Art. 513.

3. If possible, locate the incommensurable roots by the method of Art. 503.

4. If the incommensurable roots are not all located in this way, apply Sturm's Theorem (Art. 505); observing that, if the first member and its first derivative have a common factor, the given equation has multiple roots (Art. 501).

5. Approximate to the decimal portions of the incommensurable roots by Horner's method (Art. 532).

ANSWERS.

Note. In the following collection of answers, all those are omitted which, if given, would destroy the utility of the example.

Art. 40; pages 8 and 9.

1. 19.
2. 1.
3. 24.
4. $15\frac{3}{4}$.
5. $\frac{47}{12}$.
6. $\frac{9}{16}$.
7. $6\frac{4}{9}$.
8. 9.
9. $\frac{628}{135}$.
10. $\frac{1369}{144}$.
11. 48.
12. 114.
13. 24.
14. 204.
15. 310.
16. 4.
17. $\frac{9}{5}$.
18. $\frac{50}{39}$.
19. 36.
20. 48.
21. 3.
22. $\frac{35}{62}$.

Art. 43; page 12.

4. 35 and 7.
5. A, 31; B, 37.
6. A, $536; B, $664.
7. $0.93.
8. A, $20; B, $60; C, $180.
9. 52 and 73.
10. 13, 39, and 46.
11. A, $28; B, $43; C, $56.
12. Horse, $275; carriage, $100; harness, $25.
13. 15, 45, and 48.
14. 35, 70, and 105.
15. Cow, $45; sheep, $18; hog, $12.

Art. 57; page 19.

4. $2a-2b+2d$.
6. $4a^2$.
7. $2a^2-ab$.

8. $5x^3-8x^2-6$.
9. $6mn-ab-4c-x+3m^2$.
10. $x-6y$.
11. $-x+3m-7n$.
12. $-x^3+3x+2$.
13. $3a+3b+3c+3d$.
14. $5a^3-3ab^2-4b^3$.
15. a^3-4x^3.

Art. 64; pages 22 and 23.

6. $4ab$.
7. $-4abc-14x-2y-148$.
8. $2m-4y^2+12a+1$.
9. $14x^2-8y^2+5ab-7$.
10. $6b+1$.
11. $4m-8n-r+3s$.
12. $6d-2b-3a-3c$.
13. $5m^2+9n^3-71x$.
14. $-3a-2b+10c-13d+2x$.
15. $2a-b-3c$.
16. $x^4-x^3-3x^2-8x+11$.
17. $4a^3-6a^2b-2ab^2-9b^3$.
18. $4a^4-3a^3+6a^2-6a+3$.
19. $-x^2-xy$.
20. $2x-3z$.
21. $6a^4+9a^3+a^2-a-6$.
22. $-3x^3-5x^2y+8xy^2+2y^3$.
23. $5x^3+7x^2+18x$.
24. $-a^2-4ab+b^2$.
25. $5x^2+6xy-7y^2-6x-7y+6$.
26. $3x^5-5x^4-x^3-11x^2+4x-2$.
27. $4x^3+9x^2y-4xy^2-3y^3$.
28. $4a^4+7a^3-2a-4$.

Art. 69; pages 25 and 26.

4. $5x-y$.
5. $a-b+c$.
6. $5m^2-6n-4a$.
7. $-a^2-3b^2$.
8. $2a+2$.
9. $a-b+c-d+e$.
10. $a+b-c+d-e$.
11. y.
12. $14x+2$.
13. $6m-3n$.
14. $2x+4y$.
15. $a-c$.
16. $-3a-1$.
17. $4x-2$.
18. $6m+2$.
19. 0.
20. $a+2b$.

Art. 82; pages 32 to 34.

4. $15x^2-11x-14$.
5. $-12x^2+28x-15$.
6. $-6a^2+16ab-8b^2$.
7. $50x^2y^2-18$.

8. $b^3 - a^3$.
9. $10a^4b^2 - 3a^3b^3 - 18a^2b^4$.
10. $ax^4 - a$.
11. $6x^3 - 16x^2y + 6xy^2 + 4y^3$.
12. $2m^4 - 8m^3n + 18mn^3$.
13. $x^4 + 4x + 3$.
14. $30a^3 - 43a^2b + 39ab^2 - 20b^3$.
15. $8x^5 - 14x^2 - 18x + 21$.
16. $a^2 - b^2 + 2bc - c^2$.
17. $2x^4 - x^3 + 8x - 5$.
18. $6x^4 + 13x^3 - 70x^2 + 71x - 20$.
19. $6x^4 - 19x^3 + 22x + 5$.
20. $-m^5 - 37m^2 + 70m - 50$.
21. $-6x^5 - 25x^4 + 7x^3 + 81x^2 + 3x - 28$.
22. $2a^5b^2 - 3a^4b^3 - 7a^3b^4 + 4a^2b^5$.
23. $4x^{2m+7}y^3 - 16x^{m+6}y^{n+1} + 12x^5y^{2n-1}$.
24. $x^4 + x^2y^2 + y^4$.
25. $16a^4 + 4a^2b^2 + b^4$.
26. $12x^6 + 7x^4 + 5x^3 + 10x - 4$.
27. $m^6 - 3m^5n + mn^5 - 3n^6$.
28. $243x^5 - 81x^4y - 3xy^4 + y^5$.
29. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.
30. $x^3 + y^3 + z^3 - 3xyz$.
31. $6x^6 - 11x^5 + 14x^4 - 12x^3 + 6x^2 - 23x + 5$.
32. $a^2b^2 + c^2d^2 - a^2c^2 - b^2d^2$.
33. $8a^6 - 98a^4 + 152a^2 - 32$.
34. $x^3 - 6x^2 - 19x + 84$.
35. $a^6 - b^6$.
36. $36m^3 - 49m + 20$.
37. $9x^4 - 13x^2 + 4$.
38. $x^8 + x^4 + 1$.
39. $a^8 - b^8$.
40. $m^4 - 5m^2 + 4$.
41. $120x^4 + 26x^3 - 111x^2 - 14x + 24$.
42. $a^6 - 6a^4b^2 + 9a^2b^4 - 4b^6$.

Art. 83; pages 34 and 35.

2. $a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$.
3. $ab + ac - 2ad - 2bc + bd + cd$.
4. x^2.
5. $4ab + 4bc$.
6. b^2.
7. $a^4 - 2a^2b^2 + b^4$.
8. $1 - x^8$.
9. $4a + 2a^2 + 2a^3$.
10. $x^2 - 4xy + 4y^2 - 9z^2$.

11. $x^6 - y^6$. 13. $4a^2+4b^2+4c^2$. 15. $-4yz$.
12. b^2-d^2. 14. $ab+bc+ca-a^2-b^2-c^2$. 16. 0.
17. $8ac$. 18. $-9m^4+82m^2n^2-9n^4$. 19. 0.

Art. 93; pages 41 to 43.

4. $2x-5$. 7. $8-5x$. 10. $3x-1$.
5. $2+ax$. 8. $3b^3-4a^2$. 11. $4m^2-10m+7$.
6. $a-2b$. 9. $-2x^2-2ax$. 12. $9x^2-3xy+y^2$.
13. $8m^3+4m^2+2m+1$. 15. $4a^2+12ab+9b^2$.
14. $a-b-c$. 16. x^2-xy+y^2.
17. $x-3$. 19. $2x^3-3x-6$. 21. $2a+3$.
18. $4m^2-3n^2$. 20. $2x^2-7x-8$. 22. x^2-3x-y.
23. $a^6-3a^4b+9a^2b^2-27b^3$. 24. $x-y+z$.
25. $3x^3+6x^2-2x-4$.
26. $y(x^4-x^3y+x^2y^2-xy^3+y^4)$.
27. $5m^2-4m+3$. 28. $3x^2-2x+1$.
29. $3(a^4+a^3b+a^2b^2+ab^3+b^4)$.
30. $8x^2+4x+1$. 33. x^2-2x-1. 36. $2x^3-x+1$.
31. $9x^2-9x-4$. 34. $3x^2-5xy-2y^2$. 37. $3x^2+6x+9$.
32. $x^2-2xy+y^2$. 35. $a^2+3ab+5b^2$. 38. x^2-2x-3.
39. $m^3-m^2-14m+24$.
40. $x+y$. 50. $x+a$.
41. x^2y-xy^2. 51. $(b+c)a+bc$.
42. x^3-2x^2+x-2. 52. $(x+y)-3$.
43. $a^3-2a^2b-5ab^2+7b^3$. 53. $(a+b)^2-(a+b)+1$.
44. x^3-2x^2-x+1. 54. $x+a$.
45. $x+2y-3z$. 55. $(m-n)^2+2(m-n)+1$.
46. $a^n-b^m+c^r$. 56. $x^2+(a-b)x-ab$.
47. x^3+2x^2-x+1. 57. x^2-bx+c.
48. $2a^2-ab+2b^2$. 58. $a(b+c)-bc$.

Art. 96; page 46.

30. $x^2-y^2+2xz+z^2$.
31. $x^2-y^2-2yz-z^2$.
32. $1-a^2+2ab-b^2$.
33. x^4-x^2-2x-1.
34. $a^2-b^2-2ac+c^2$.
35. a^4-2a^2+1.
36. $x^4-4x^2+12x-9$.
37. $m^4+m^2n^2+n^4$.

Art. 111; pages 56 and 57.

26. $(x+y+2)(x+y-2)$.
27. $(a-b+c)(a-b-c)$.
28. $(a+b-c)(a-b+c)$.
29. $(a+b+c)(a-b-c)$.
30. $(c+d+1)(c+d-1)$.
31. $(3+x-y)(3-x+y)$.
32. $(2m^2+2b-1)(2m^2-2b+1)$.
33. $(2a-b+3d)(2a-b-3d)$.
34. $(a+b-m+n)(a-b-m-n)$.
35. $(x+y-c-d)(x-y-c+d)$.
36. $(a+b+m-n)(a-b+m+n)$.
37. $(a+b+c+d)(a-b+c-d)$.

Art. 120; page 64.

27. $2xy(x+y+z)(x+y-z)$.
40. $(x+y+z)(x+y-z)(x-y+z)(x-y-z)$.
46. $(a+b)(a-b)(x+y)(x-y)$.
50. $(x-2)(x+3)(x^2-x+6)$.
51. $(a-1)(a-2)(a+4)(a+5)$.
53. $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$.
55. $(x-1)^2(x+1)$.
56. $2(a-b)(a+2b)$.
58. $(x+2)^2(x-2)^2$.
59. $(x-y)^3$.
60. $(a-1)(a+2)(a-2)(a+3)$.

Art. 131; pages 73 and 74.

1. $x-2$.
2. $2x+3$.
3. $a-1$.
4. $8x-7a$.
5. $2a-5$.
6. x^2-mx.
7. $5m+3$.
8. $2a-3x$.
9. $x-2$.
10. $2x-1$.
11. $2m-5n$.
12. $x+2$.

13. $ax^2 - ax$. 15. $a^2 - a - 1$. 17. $3x + 2$. 19. $x + 1$.
14. $x^2 + x + 1$. 16. $x^3 - 2x$. 18. $a - x$. 20. $x - 2y$.
21. $2x - 3$. 22. $3a + b$.

Art. 132; page 74.

1. $2x + 7$. 2. $2x - 5$. 3. $3m + 2n$. 4. $3a - 1$.
5. $x + 4$. 6. $x - 1$. 7. $2a + 3$.

Art. 136; page 76.

4. $270m^2n^2$. 7. $336x^2y^3z$. 9. $480m^3n^2x^2y^2$.
5. $210abc$. 8. $360a^2c^2d^3$. 10. $252x^3y^3z^3$.
11. $1080a^2b^3c^3d^4$.

Art. 137; pages 76 and 77.

2. $y(x^2 - y^2)$.
3. $(x^2 - 1)(x - 8)$.
4. $24ab(a^2 - b^2)$.
5. $(m + n)(m^3 - n^3)$.
6. $a^2 - 4ab + 3b^2$.
7. $xy(x + y)(x - y)^2$.
8. $12abc(a^2 - b^2)$.
9. $(x+3a)(x-5a)(x+7a)$.
10. $n(m^2 - n^2)$.
11. $(a + b)(a - 3b)(x - 2)$.
12. $ax(x + a)(x^3 - a^3)$.
13. $24(a + b)^2(a - b)^2$.
14. $ax(x-3)(x-7)(x+8)$.
15. $x^4 - 2x^2 + 1$.
16. $24(1 - x^4)$.
17. $(x + 1)(x - 2)(x + 3)(x + 4)$.
19. $(2m + 1)(2m - 1)^2(4m^2 + 2m + 1)$.
20. $a^3(a - 1)(a^3 + 1)$.
21. $(a - 1)(a - 3)(a + 4)(a - 5)$.
22. $(1 + x)^2(1 - x)^2(1 + x^2)^2$.
23. $(a + b + c)(a + b - c)(a - b - c)$.
24. $3ab(a - b)(x - y)^2$.
25. $2ax^2(3x + 2)^2(9x^2 - 6x + 4)$.
26. $(x + y + z)(x + y - z)(x - y + z)$.

Art. 139; page 79.

2. $4x^3-13x+6$.
4. $24x^3+26x^2-219x-56$.
3. $24x^3+22x^2-177x+140$.
5. $2ax(6x^3+x^2-42x-45)$.
6. $a^4-16a^3b+86a^2b^2-176ab^3+105b^4$.
7. $2n(2m^5-5m^4+3m^3-5m^2+4m+4)$.
8. $a(30x^3-11ax^2-59a^2x+12a^3)$.
9. $2a^4+a^3-17a^2-4a+6$.
10. $2x^5+3x^4-4x^3+5x-6$.
11. $a^5-a^3b^2+a^2b^3-b^5$.
12. $ax(x^5+x^4-x^3-3x^2-3x-1)$.
13. $x(6x^5-31x^4-4x^3+44x^2+7x-10)$.
14. $x^6+2x^5-4x^4-7x^3-16x^2+32x-8$.

Art. 140; page 79.

1. $4x^4-4x^3-39x^2+4x+35$.
2. $18a^4-33a^3+14a^2+3a-2$.
3. $20x^4-24x^3-51x^2+41x-6$.
4. $2x^2(12x^4-32x^3-29x^2+57x-18)$.
5. $a^5+3a^4-23a^3-27a^2+166a-120$.

Art. 149; pages 83 and 84.

12. $\frac{cd}{3xy}$.
13. $\frac{x^3}{2y^2}$.
14. $\frac{2x^2y}{x-5}$.
15. $\frac{a-5}{a+7}$.
16. $\frac{2a^2+ab}{ab+2b^2}$.
17. $\frac{2c-5}{c(2c+5)}$.
18. $\frac{m-2}{m+9}$.
19. $\frac{a(3n+2)}{b(3n-2)}$.
20. $\frac{a+2b}{a+3b}$.
21. $\frac{2x+y}{x}$.
22. $\frac{c-d}{a^2+ab+b^2}$.
23. $\frac{a(x+2)}{x(x-7)}$.
24. $\frac{9y^2+15y+25}{3y-5}$.
25. $\frac{x-1}{2x+1}$.
26. $\frac{1}{ax(x+4)}$.
27. $\frac{a+b+c}{a-b+c}$.
28. 1.
29. $\frac{a-b+c-d}{a+b-c-d}$.

Art. 150; page 85.

2. $\dfrac{x-5}{3x+7}$. 4. $\dfrac{m-1}{6m-5}$. 6. $\dfrac{6m-n}{5m-7n}$. 8. $\dfrac{3x-2}{x+3}$.

3. $\dfrac{5a+7}{a-2}$. 5. $\dfrac{x+1}{x^2+x+1}$. 7. $\dfrac{x+2}{x-3}$. 9. $\dfrac{2y-3}{2y-5}$.

10. $\dfrac{a^2-a-1}{2a^2+a+1}$. 11. $\dfrac{x^2-3xy+y^2}{x^2-xy+3y^2}$.

Art. 155; pages 87 and 88.

5. $x^2-xy+y^2+\dfrac{y^3}{x+y}$.

6. $2x+6-\dfrac{23}{x-3}$.

7. $a-2+\dfrac{2a-4}{a^2+a-1}$.

8. $3x-\dfrac{5x-7}{4x-1}$.

10. $2m^2-5mn+7n^2-\dfrac{n^3}{2m-3n}$.

11. $a^2-3-\dfrac{3a-5}{2a^2-a-3}$.

12. $x+1+\dfrac{x+3}{x^2+x+1}$.

14. $2x-3-\dfrac{2x+3}{3x^2-2x+1}$.

Art. 156; page 89.

3. $\dfrac{(x+1)^2}{x}$.

4. $\dfrac{x^2+4x-1}{x+3}$.

5. $\dfrac{5m^2-2mn-4n^2}{3m+n}$.

6. $\dfrac{3x-4}{8}$.

7. $\dfrac{2n}{m+n}$.

8. $\dfrac{2ab}{a+b}$.

9. $\dfrac{6x^2-x}{2x+1}$.

10. $\dfrac{a^3+b^3}{a-b}$.

11. $\dfrac{2y}{x-y}$.

12. $\dfrac{2m^2}{m+n}$.

13. $\dfrac{a^3-b^3}{a+b}$.

14. $\dfrac{x^3-3x^2}{x-2}$.

15. $\dfrac{2n^3}{m^2+mn+n^2}$.

16. $\dfrac{9x^3}{2x-1}$.

17. $\dfrac{6xy}{2y-x}$.

18. $\dfrac{x^4-5x^2-1}{x^2+3x-2}$.

Art. 157; page 91.

6. $\dfrac{2a^2+4a}{(a+3)(a^2-4)}$, $\dfrac{4a^2+12a}{(a+3)(a^2-4)}$.

7. $\dfrac{x^2+x+1}{(x+1)(x^3-1)}$, $\dfrac{x+1}{(x+1)(x^3-1)}$.

8. $\dfrac{mn(m^2-n^2)}{n(m^2-n^2)}$; $\dfrac{m^3(m+n)}{n(m^2-n^2)}$, $\dfrac{mn^3}{n(m^2-n^2)}$.

9. $\dfrac{2(a^3+a^2b+ab^2+b^3)}{a^4-b^4}$, $\dfrac{3(a^3-a^2b+ab^2-b^3)}{a^4-b^4}$, $\dfrac{4(a^2-b^2)}{a^4-b^4}$.

11. $\dfrac{2a^2b}{2a(a-b)(m+n)}$, $\dfrac{m^2-n^2}{2a(a-b)(m+n)}$.

12. $\dfrac{x^2-9}{(x-1)(x-2)(x-3)}$, $\dfrac{x^2-1}{(x-1)(x-2)(x-3)}$, $\dfrac{x^2-4}{(x-1)(x-2)(x-3)}$.

Art. 158; pages 92 to 97.

3. $\dfrac{12x+7}{36}$. 6. $\dfrac{3m^2n^2-4}{6m^2n^3}$. 9. $\dfrac{4ab-b-4a^3}{12a^3b}$.

4. $\dfrac{6a-5b}{10a^2b^2}$. 7. $\dfrac{5b^2+4a^2}{120ab}$. 10. $\dfrac{1}{15}$. 11. $\dfrac{m}{42}$

5. $-\dfrac{a+3}{24}$. 8. $\dfrac{5a+b}{24}$. 12. $\dfrac{3x-2}{18x^2}$.

13. $-\dfrac{1}{60}$. 14. $\dfrac{4bcd+6acd-3abd-2abc}{48abcd}$.

17. $\dfrac{2}{1-x^2}$. 20. $\dfrac{a^2+b^2}{a^2-b^2}$. 23. $\dfrac{4x}{1-x^2}$.

18. $\dfrac{5}{6+x-x^2}$. 21. $\dfrac{4ab}{a^2-b^2}$. 24. $\dfrac{3m^2+n^2}{(m+n)(m-n)^2}$.

19. $\dfrac{1}{x^2+15x+56}$. 22. $\dfrac{y}{x+y}$. 25. $\dfrac{5}{(a+3)(a-2)^2}$.

26. $\dfrac{4x}{x+y}$.

27. $\dfrac{a+b}{a-b}$.

28. 0.

29. $\dfrac{4x^2}{x^2-y^2}$.

30. $-\dfrac{2}{x(4x^2-1)}$.

31. $\dfrac{4ab^2}{a^4-b^4}$.

32. $\dfrac{1+x+x^2}{(1-x)^3}$.

33. 0.

34. $\dfrac{4x}{x^2+3x-10}$.

35. 2.

36. $\dfrac{x^2-2}{x(x^2-1)}$.

37. 0.

38. $\dfrac{(x+2)^2}{(x+1)(x^3-1)}$.

40. $-\dfrac{2(x-z)}{(x+y)(y+z)}$.

39. $\dfrac{13-18x}{(x+1)(x+2)(x-3)}$.

43. $\dfrac{4b-3a}{ab(a-b)}$.

44. $\dfrac{19a-1}{6(a-1)}$.

45. $\dfrac{3}{9x-x^3}$.

46. $\dfrac{2m+n}{m(m^2-n^2)}$.

47. $\dfrac{1}{(x+2)(x+a)}$.

48. $\dfrac{2a}{a+b}$.

49. $\dfrac{x^2}{x^2-1}$.

50. $\dfrac{2x}{x^2-5x+6}$.

51. 0.

52. $-\dfrac{5}{(x-2)(x-3)(x-4)}$.

Art. 159; pages 98 and 99.

4. $\dfrac{1}{a}$.

5. $\dfrac{1}{2}$.

6. $\dfrac{z}{4xy}$.

7. a^3.

8. $\dfrac{n}{10}$.

9. $\dfrac{3x-1}{x-2}$.

10. $\dfrac{x^2-x-20}{x^2}$.

11. $\dfrac{a-2b}{a^2}$.

12. $\dfrac{ab-b^2}{ax+bx}$.

13. $\dfrac{xy}{x^2-y^2}$.

14. $\dfrac{ax-2a}{a+1}$.

15. $\dfrac{3x+15}{x^2+2x}$.

16. $\dfrac{1+y}{x}$.

17. $\dfrac{x}{x-2y}$.

18. $\dfrac{x^2-2x}{x-y+z}$.

19. x^2+xy.

20. 1.

Art. 160; pages 100 and 101.

3. $\dfrac{a^2}{10b^3m^2n^3}$.

4. $\dfrac{3ny^3}{5mx}$.

5. $\dfrac{a+6}{a+4}$.

6. $\frac{3x-12}{x^3}$. 8. $\frac{(a-b)^2}{ab+b^2}$. 10. $\frac{a+2}{a^2-2a}$.

7. $\frac{x^2+x-20}{x-2}$. 9. $\frac{m^2-mn+n^2}{m^2-mn}$. 11. $\frac{3x-2y}{x+y}$.

12. $\frac{a-2b}{a+b}$. 13. $\frac{2x-3y}{2x+3y}$.

Art. 161; pages 102 and 103.

4. $x-1$. 10. $\frac{m+n}{n}$. 16. $\frac{a-b-c}{a+b-c}$.

5. $\frac{1}{a+b}$. 11. $\frac{x+5}{x+1}$. 17. $\frac{1}{1+x^2}$.

6. x^2-x+1. 12. $\frac{5x-16}{3x-10}$. 18. $\frac{ac-bd}{ad+bc}$.

7. $a-1$. 13. $\frac{a-b}{a}$. 19. $\frac{4}{3x+3}$.

8. x^2-2xy. 14. $\frac{x^2-y^2}{xy}$. 20. 1.

9. $\frac{x-4}{x+6}$. 15. x. 21. $\frac{x-a}{x+2a}$.

22. $\frac{ab}{a^2+b^2}$. 23. $\frac{2(m-n)}{(m+n)^2}$.

Art. 162; pages 104 and 105.

1. $\frac{bx-a}{x^2}$. 2. $\frac{m-1}{3m-15}$. 3. $\frac{x^2}{(1+x)^4}$.

4. $\frac{ab+b^2}{am+an}$. 5. $\frac{x^3-1}{2(1+x)(1+x^2)}$.

6. $\frac{2(a+b)}{a^2}$. 9. $\frac{a}{b}-\frac{b}{a}$. 12. $\frac{5x^2}{3-3x^2}$.

7. $x^2+1+\frac{1}{x^2}$. 10. $\frac{1}{1+x^2}$. 13. $\frac{2n-4}{3n}$.

8. $\frac{1}{1+x^2}$. 11. $\frac{b}{ax+b}$. 14. x^2+xy.

15. $\frac{4b}{a+b}$.

16. $x^4-\frac{1}{x^4}$.

17. $\frac{4x^2-2}{4x^4+1}$.

18. $\frac{2ax^2}{x^4-a^4}$.

19. $\frac{(x^2+y^2)^2}{x^4+y^4}$.

20. $\frac{x-2}{x-5}$.

21. $\frac{a^2(a-6b)}{a-3b}$.

22. $\frac{y^3}{x^2-y^2}$.

23. 3.

24. $\frac{x^2+4}{(x-2)^3}$.

25. $\frac{a^2+b^2}{a}$.

26. $\frac{3x+3y}{x-7y}$.

27. $\frac{ab+bc+ca}{(a+b)(b+c)(c+a)}$.

28. $\frac{3}{1+9x}$.

29. $\frac{2x^2-2}{x^4+x^2+1}$.

30. $-\frac{a^4+a^2b^2+b^4}{ab(a-b)^2}$.

Art. 174; pages 109 and 110.

3. 14. 4. -5. 5. 3. 6. -3. 7. -8. 8. 2. 9. $\frac{3}{8}$. 10. $-1\frac{7}{8}$. 11. 1. 12. $\frac{4}{3}$. 13. 2. 14. $-\frac{2}{3}$. 15. -2. 16. 16. 18. $-\frac{3}{2}$. 19. 0. 20. 1. 21. 2. 22. 2. 23. -4. 24. 2. 25. 1. 26. 2. 27. -8. 28. $-\frac{4}{3}$. 29. 4. 30. -3.

Art. 175; pages 111 to 115.

2. -6. 3. $\frac{2}{3}$. 4. $\frac{2}{3}$. 5. 3. 6. 5. 7. $-1\frac{2}{3}$. 8. $\frac{2}{7}$. 9. $-2\frac{4}{9}$. 10. 56. 11. 2. 13. -2. 14. $\frac{1}{3}$. 15. 5. 16. -3. 17. $\frac{1}{2}$. 18. 7. 19. $-1\frac{8}{11}$. 20. -5. 21. 4. 22. -5. 23. -2. 24. $\frac{2}{3}$. 25. $-\frac{1}{2}$. 26. $-\frac{2}{3}$. 27. 1.

30. 7.	**34.** -7.	**38.** 5.	**42.** -4.	**46.** $\frac{1}{8}$.
31. $-1\frac{1}{14}$.	**35.** -2.	**39.** 3.	**43.** -7.	**47.** $\frac{2}{5}$.
32. 2.	**36.** $-\frac{3}{2}$.	**40.** 0.	**44.** $-2\frac{2}{5}$.	**48.** -1
33. $-\frac{1}{2}$.	**37.** $-\frac{4}{3}$.	**41.** 1.	**45.** $-1\frac{5}{9}$.	**49.** $\frac{7}{2}$.

Art. 176; pages 116 and 117.

3. $\frac{3c-d}{2a+b}$.	**8.** $\frac{2m^2}{3n}$.	**13.** $3a-3$.	**18.** mn.
4. $\frac{5a}{2b}$.	**9.** $a-b$.	**14.** $-\frac{n}{2}$.	**19.** $7a$.
5. $\frac{a+1}{a-1}$.	**10.** $a+b$.	**15.** $-\frac{a}{b}$.	**20.** $\frac{a}{3}$.
6. $8b+4a$.	**11.** $\frac{a-b}{2}$.	**16.** $12a^3$.	**21.** $-\frac{a}{3b}$.
7. $\frac{4a^2}{5}$.	**12.** $b-2c$.	**17.** $\frac{1}{a+2}$.	**22.** $\frac{3b}{a}$.

23. $\frac{1}{2(a+b)}$. **24.** n.

Art. 177; page 118.

2. .8.	**4.** 2.	**6.** .7.	**8.** 8.	**10.** 0.
3. -3.	**5.** 50.	**7.** 5.	**9.** $-.04$.	

Art. 179; pages 120 to 131.

4. 30.	**7.** 35, 24.	**10.** A, \$30; B, \$60.
5. 20, 14.	**8.** A, 60; B, 15.	**11.** 116, 91.
6. 120.	**9.** 23.	**12.** 7 and 10.

13. 12 oxen, 24 cows. **14.** 47, 33.

15. Wife, \$864; a daughter, \$288; a son, \$144.

16. A, \$18; B, \$48; C, \$4.

17. Infantry, 2450; cavalry, 196; artillery, 98.

18. A, 62; B, 28.

19. On foot, 880; by water, 1540; on horseback, 616.

21. $\dfrac{a}{1+mn}$, $\dfrac{amn}{1+mn}$. **22.** 22, 23, 24, 25. **23.** 29, 14

24. A, \$14; B, \$13; C, \$11; D, \$9. **25.** 115.

26. 120, 60, 20, 5. **27.** A, 59; B, 23. **28.** 22, 23. **29.** 36.

30. A, $\dfrac{am(n-1)}{m-n}$; B, $\dfrac{a(n-1)}{m-n}$. **33.** $8\frac{2}{11}$.

31. $\dfrac{an^2}{n^2+n+1}$, $\dfrac{an}{n^2+n+1}$, $\dfrac{a}{n^2+n+1}$. **34.** $\dfrac{11}{12}$.

35. 144 sq. yds. **36.** A, \$35; B, \$38. **37.** $\dfrac{ab}{a+b}$.

38. $\dfrac{abc}{ab+bc+ca}$. **40.** 2 dollars, 20 dimes, 4 cents.

41. 17 two-penny pieces, 36 farthings. **42.** \$2.75.

43. Worked, 20; absent, 16. **44.** \$58. **45.** $\dfrac{a-bc}{c+1}$.

46. 48 minutes. **48.** 84. **49.** 93.

50. 30 bushels at 9 shillings; 10 at 13 shillings.

51. Gold, 3377 oz.; silver, 783 oz. **52.** 36. **54.** $8\frac{3}{4}$ miles.

55. $\dfrac{an}{b-a}$. **56.** A, 12 miles; B, 14 miles.

57. \$1200 in 5 per cents; \$2000 in 6 per cents.

58. $\dfrac{100a}{rt+100}$. **59.** $\dfrac{100(a-p)}{pr}$. **61.** 28, 9.

62. 82, 31. **63.** 12,121 men; 110 on a side at first.

64. $\dfrac{a-c}{b+1}$, $\dfrac{ab+c}{b+1}$. **65.** $\dfrac{7}{13}$. **66.** 51. **67.** $\dfrac{100(a-p)}{pt}$.

68. Picture, \$5.28; frame, \$3.96. **69.** $\dfrac{3}{4}$.

71. $5\frac{5}{11}$ minutes after 7. **73.** $27\frac{3}{11}$ minutes after 5.

72. $43\frac{7}{11}$ minutes after 2. **74.** $5\frac{5}{11}$ minutes after 1.

75. 7.

76. 7 dimes, 13 half-dimes.

77. $\frac{abc}{b+c}$ miles.

78. $16\frac{4}{11}$ minutes after 6.

79. $5\frac{5}{11}$ minutes, or $38\frac{2}{11}$ minutes after 10.

80. First kind, $\frac{a(c-b)}{a-b}$; second, $\frac{b(a-c)}{a-b}$.

81. 10 A.M.

82. $27\frac{3}{11}$ minutes after 4.

83. 48.

84. \$6480.

85. Horse, \$180; carriage, \$95.

86. A, 52 miles; B, 55 miles.

87. $\frac{am+bn+cp}{a+b+c}$ cents.

88. A, 11 days; B, 22 days; C, 33 days.

89. A, \$750; B, \$500.

90. 48 minutes after 9.

91. A, 3 days; B, 4 days; C, 5 days.

92. 18 yards, 15 yards.

93. $\frac{b+d}{a-c}$.

94. \$2000.

95. \$1400.

96. \$15,000.

97. $16\frac{4}{11}$ minutes after 8.

98. Greyhound, 72; fox, 108.

Art. 184; pages 134 and 135.

3. $x=5$, $y=-2$.

4. $x=4$, $y=3$.

5. $x=5$, $y=2$.

6. $x=12$, $y=7$.

7. $x=-3$, $y=6$.

8. $x=7$, $y=-1$.

9. $x=\frac{1}{2}$, $y=-3$.

10. $x=-\frac{3}{4}$, $y=\frac{2}{5}$.

11. $x=-4$, $y=-5$.

12. $x=-2$, $y=4$.

13. $x=-\frac{4}{3}$, $y=-2$.

14. $x=-\frac{2}{3}$, $y=\frac{3}{2}$.

15. $x=8$, $y=10$.

16. $x=-1$, $y=-3$.

17. $x=-\frac{2}{3}$, $y=-\frac{5}{7}$

Art. 185; page 136.

2. $x=5$, $y=2$.

3. $x=3$, $y=-1$.

4. $x=-2$, $y=-2$.

5. $x=1$, $y=-\frac{1}{2}$.

6. $x=-\frac{1}{3}$, $y=\frac{1}{4}$.

7. $x=-3$, $y=-2$.

8. $x=-1$, $y=-2$.

9. $x=\frac{1}{5}$, $y=-1$.

10. $x=\frac{1}{2}$, $y=2$.

11. $x=\frac{9}{5}$, $y=-\frac{3}{5}$.

12. $x=-\frac{1}{4}$, $y=-3$.

13. $x=-2$, $y=19$.

Art. 186; page 137.

2. $x=2$, $y=3$.

3. $x=-\frac{1}{2}$, $y=-1$.

4. $x=5$, $y=-2$.

5. $x=-2$, $y=3$.

6. $x=12$, $y=-3$.

7. $x=-1$, $y=4$.

8. $x=4$, $y=\frac{2}{3}$.

9. $x=-3$, $y=-1\frac{1}{4}$.

10. $x=-\frac{1}{5}$, $y=-\frac{2}{3}$.

11. $x=-\frac{1}{4}$, $y=2$.

12. $x=1$, $y=-\frac{1}{3}$.

13. $x=-\frac{2}{3}$, $y=\frac{1}{2}$.

Art. 187; pages 138 to 143.

2. $x=10$, $y=5$.

3. $x=24$, $y=-18$.

4. $x=-16$, $y=-12$.

5. $x=\frac{3}{2}$, $y=-\frac{4}{3}$.

6. $x=2$, $y=3$.

7. $x=-.3$, $y=.08$.

8. $x=18$, $y=6$.

9. $x=4$, $y=9$.

10. $x=-2$, $y=3$.

11. $x=-2$, $y=-4$.

12. $x=2$, $y=\frac{1}{2}$.

13. $x=60$, $y=40$.

14. $x=1$, $y=-2$.

15. $x=3$, $y=-4$.

16. $x=\frac{1}{3}$, $y=-\frac{3}{2}$.

17. $x=3$, $y=-2$.

18. $x=-3$, $y=11$.

19. $x=\frac{4}{3}$, $y=\frac{15}{4}$.

20. $x=12$, $y=6$.

21. $x=1$, $y=5$.

22. $x=-6$, $y=-5$.

23. $x=2$, $y=-2$.

24. $x=13$, $y=3$.

25. $x=-\frac{39}{4}$, $y=-\frac{33}{8}$.

27. $x=\frac{4a+3b}{17}$, $y=\frac{2b-3a}{17}$.

28. $x=\frac{dm-bn}{ad-bc}$, $y=\frac{an-cm}{ad-bc}$.

29. $x=\frac{c-bd}{a-b}$, $y=\frac{c-ad}{a-b}$.

30. $x=\frac{bp}{an+bm}$, $y=\frac{ap}{an+bm}$.

31. $x=\frac{dm+bn}{ad+bc}$, $y=\frac{cm-an}{ad+bc}$.

32. $x=\frac{n'p-np'}{mn'-m'n}$, $y=\frac{m'p-mp'}{mn'-m'n}$.

33. $x=\frac{ac(bm+dn)}{ad+bc}$, $y=\frac{bd(cn-am)}{ad+bc}$.

34. $x=ab$, $y=a+b$.

35. $x=\frac{1}{a}$, $y=\frac{1}{b}$.

36. $x=a^2b$, $y=ab^2$.

37. $x=\frac{m}{n^2}$, $y=\frac{n}{m^2}$.

38. $x=a+b$, $y=a-b$.

39. $x=a$, $y=b$.

40. $x=\frac{a^2}{m+n}$, $y=\frac{m^2-n^2}{a}$.

41. $x=\frac{1}{2a}$, $y=\frac{1}{2a}$.

42. $x=(a+b)^2$, $y=(a-b)^2$.

44. $x=4$, $y=2$.

45. $x=-5$, $y=3$.

46. $x=-2$, $y=-1$.

47. $x=\frac{1}{2}$, $y=-\frac{1}{4}$.

48. $x=m+n$, $y=m+n$.

49. $x=\frac{bc-ad}{bn-dm}$, $y=\frac{bc-ad}{cm-an}$.

50. $x=\frac{2}{3}$, $y=\frac{3}{4}$.

51. $x=\frac{1}{n}$, $y=\frac{1}{m}$.

Art. 189; pages 145 to 147.

3. $x=3$, $y=-1$, $z=0$.

4. $x=-2$, $y=3$, $z=1$.

5. $x=1$, $y=2$, $z=-4$.

6. $x=2$, $y=-1$, $z=-2$.

7. $x=23$, $y=6$, $z=24$.

8. $x=-2$, $y=3$, $z=7$.

9. $x=-4$, $y=2$, $z=-5$.

10. $x=-7$, $y=-2$, $z=1$.

11. $x=8$, $y=-3$, $z=-4$.

12. $x=-\frac{1}{2}$, $y=\frac{3}{2}$, $z=-\frac{5}{2}$.

13. $x=-5$, $y=-5$, $z=-5$.

14. $x=3$, $y=4$, $z=6$.

15. $x=4$, $y=6$, $z=2$.

16. $x=10$, $y=2$, $z=3$.

17. $x=-24$, $y=-48$, $z=60$.

18. $u=\frac{1}{4}$, $x=-\frac{1}{6}$, $y=\frac{1}{8}$, $z=-\frac{1}{10}$.

19. $u=-7$, $x=3$, $y=-5$, $z=1$.

20. $x=\frac{4}{3}$, $y=4$, $z=\frac{4}{5}$.

21. $x=-\frac{1}{3}$, $y=\frac{1}{2}$, $z=\frac{1}{4}$.

22. $x=\frac{1}{a}$, $y=\frac{1}{a^2}$, $z=\frac{1}{a^3}$.

23. $u=4$, $x=5$, $y=6$, $z=7$.

24. $x=\frac{1}{2}$, $y=-\frac{1}{3}$, $z=-1$.

25. $x=7$, $y=-3$, $z=-5$.

26. $x=\dfrac{b^2+c^2-a^2}{2bc}$, $y=\dfrac{c^2+a^2-b^2}{2ca}$, $z=\dfrac{a^2+b^2-c^2}{2ab}$.

27. $x=1\frac{2}{3}$, $y=1$, $z=\frac{1}{2}$.

28. $x=3$, $y=-2$, $z=-1$.

29. $x=a$, $y=a$, $z=1$.

30. $x=abc$, $y=ab+bc+ca$, $z=a+b+c$.

Art. 190; pages 149 to 157.

3. 32, 18. **5.** A, 30; B, 20. **7.** 24 and 18.

4. 14, 7. **6.** $\frac{4}{15}$. **8.** A, \$96; B, \$48.

9. A, 48; B, 18. **10.** $\frac{7}{19}$. **11.** A, 16 days; B, $26\frac{2}{3}$ days.

12. 38, 13. **13.** $\frac{4}{5}$. **14.** $\frac{a^3+a}{a^2-a+1}$.

15. Better horse, \$160; poorer horse, \$100; harness, \$40.

16. First, 8 cents; second, 7 cents; third, 4 cents.

18. 30 cents; 15 oranges.

19. $13\frac{1}{3}$ bushels at 60 cents; $26\frac{2}{3}$ at 90 cents.

20. Income tax, \$20; assessed tax, \$30.

21. 120, at 7 cents each. **24.** 10, 22, 26.

22. Length, 80 ft.; width, 60 ft. **25.** 42, 38, 32, 24.

23. A, \$45; B, \$55. **27.** 74. **28.** 326.

29. A, $9\frac{3}{5}$ days; B, 16 days; C, 48 days.

30. Length, 30 rods; width, 20 rods; area, 600 sq. rods.

31. $\frac{b+c-a}{2}$, $\frac{c+a-b}{2}$, $\frac{a+b-c}{2}$. **32.** 246.

33. Number of persons, $\frac{(a+b)mn}{bm-an}$;

each received $\frac{ab(m+n)}{bm-an}$ dollars.

34. Whole sum, \$1200; eldest, \$400; second, \$300; third, \$240; fourth, \$260.

35. 30 at 2 for 5 cents; 36 at 3 for 8 cents.

36. 42. **37.** 38.

38. A, $\frac{2mnp}{mn+np-mp}$ days; B, $\frac{2mnp}{mp+np-mn}$;

C, $\frac{2mnp}{mp+mn-np}$.

40. Current, $\frac{ad-bc}{2bd}$ miles an hour; crew, $\frac{ad+bc}{2bd}$.

41. Going, 4 hours; returning, 6 hours.

42. 759. **43.** First, 22; second, 10.

44. 65. **45.** First rate, 6 p.c.; second, 5 p.c.

46. \$120 at 5 per cent.

47. $\frac{bm-an}{n-n}$ dollars at $\frac{100(a-b)}{bm-an}$ per cent.

49. 15 miles; $5\frac{1}{2}$ miles an hour. **50.** 43.

51. 40 miles an hour. **53.** A, 8; B, 6.

52. A, \$13; B, \$7; C, \$4. **54.** 58, 43, 14.

55. A, \$7; B, \$22; C, \$21; D, \$16.

56. A, \$78; B, \$42; C, \$24.

57. A, 8 hrs.; B, 9 hrs.; C, 12 hrs.

58. \$2000 in $3\frac{1}{2}$ per cents; \$1600 in four per cents.

59. A, 16; B, 12. **60.** Fore-wheel, 8 ft.; hind-wheel, 12 ft.

61. A, 8 days: B, 12 days. **62.** A, 6; B, 5.

Art. 194; page 160.

4. $4x^4+4x^3+5x^2+2x+1$. **6.** $x^4+8x^3+12x^2-16x+4$.

5. $x^4-6x^3+11x^2-6x+1$. **7.** $4x^4-4x^3-11x^2+6x+9$.

8. $9a^4-30a^3+49a^2-40a+16$.

9. $4x^4+20x^3-3x^2-70x+49$.

11. $x^6-4x^4+10x^3+4x^2-20x+25$.

12. $4x^6+12x^5+9x^4+4x^3+6x^2+1$.

13. $9a^4-12a^3b-26a^2b^2+20ab^3+25b^4$.

14. $16m^4+8m^3n^2-23m^2n^4-6mn^6+9n^8$.

17. $1+2x+3x^2+4x^3+3x^4+2x^5+x^6$.

18. $9x^6-12x^5-2x^4+28x^3-15x^2-8x+16$.

19. $x^6-8x^5+12x^4+10x^3+28x^2+12x+9$.

Art. 195; page 162

3. $x^3+9x^2+27x+27$. 6. $a^3+12a^2b+48ab^2+64b^3$.

4. $8x^3-12x^2+6x-1$. 7. $27m^6-27m^4+9m^2-1$.

9. $a^3+15a^2b+75ab^2+125b^3$.

10. $8x^3-60x^2y+150xy^2-125y^3$.

11. $8x^9-36x^7+54x^5-27x^3$.

12. $216x^6+108x^5y+18x^4y^2+x^3y^3$.

13. $27m^3+135m^2n+225mn^2+125n^3$.

14. $27x^3y^3-108a^2x^2y^2+144a^4xy-64a^6$.

16. $x^6-3x^5+5x^3-3x-1$.

17. $a^3-3a^2b+3a^2+3ab^2-6ab+3a-b^3+3b^2-3b+1$.

18. $a^3+3a^2b-3a^2c+3ab^2-6abc+3ac^2+b^3-3b^2c+3bc^2-c^3$.

19. $x^6-6x^5+18x^4-32x^3+36x^2-24x+8$.

20. $x^6+9x^5+30x^4+45x^3+30x^2+9x+1$.

21. $8x^6-36x^5+42x^4+9x^3-21x^2-9x-1$.

Art. 196; page 164.

10. $16+32x+24x^2+8x^3+x^4$.

11. $x^4-16x^3+96x^2-256x+256$.

12. $a^5-15a^4+90a^3-270a^2+405a-243$.

13. $a^5+10a^4+40a^3+80a^2+80a+32$.

14. $x^6-12x^5+60x^4-160x^3+240x^2-192x+64$.

16. $a^5-15a^4x+90a^3x^2-270a^2x^3+405ax^4-243x^5$.

17. $81+216b+216b^2+96b^3+16b^4$.

19. $x^{12}-16x^9+96x^6-256x^3+256$.

20. $64a^{12}+192a^{10}b+240a^8b^2+160a^6b^3+60a^4b^4+12a^2b^5+b^6$.

21. $16m^{12}-96m^9n^2+216m^6n^4-216m^3n^6+81n^8$.

Art. 204; page 169.

3. a^2-2a+1. 6. $5+3x+x^2$. 9. $2a^2-5ab+8b^2$.

4. $2x^2-x-1$. 7. $3x^2-4x-5$. 10. $1-7x-2x^2$.

5. $3-2x+x^2$. 8. $m+1-\frac{1}{m}$. 11. $a-b-c$.

12. $x-2y+3z.$ 13. $3x^3+5x^2-7.$ 14. $4c^3-5c-3.$

15. $a^3-2a^2+5a+3.$

16. $2x^3-x^2y-xy^2-2y^3.$

17. $2x^3-3x^2+4x-5.$

18. $\frac{a^2}{3}-ab+\frac{b^2}{2}.$

19. $3x^3-2x^2y+xy^2-2y^3.$

20. $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\cdots$

21. $1-a-\frac{a^2}{2}-\frac{a^3}{2}-\cdots$

22. $a-2b-\frac{3b^2}{2a}-\frac{3b^3}{a^2}-\cdots$

23. $2x+\frac{y}{2x}-\frac{y^2}{16x^3}+\frac{y^3}{64x^5}-\cdots$

Art. 207; pages 172 and 173.

2.	214.	**10.**	21.12.	**19.**	5.5678.	**27.**	1.3229.
3.	523.	**11.**	.04738.	**20.**	4.1593.	**28.**	.43301.
4.	809.	**12.**	900.8.	**21.**	.83666.	**29.**	1.0541.
5.	5.76.	**13.**	.8253.	**22.**	.28284.	**30.**	.44721.
6.	.497.	**15.**	1.4142.	**23.**	.37947.	**31.**	1.1547.
7.	.286.	**16.**	1.7321.	**24.**	.031623.	**32.**	.64550.
8.	.722.	**17.**	2.2361.	**25.**	.079057.	**33.**	1.1726.
9.	1.082.	**18.**	3.3166.	**26.**	1.4444.	**34.**	.94868.
		35.	.62361.	**36.**	.42492.		

Art. 209; page 175.

7. $x^2+2x-4.$ 10. $2x^2-3x-1.$ 13. $x^2+xy-2y^2.$

8. $y^2-y-1.$ 11. $2a^2-a-5.$ 14. $3a^2-2ab-b^2.$

9. $x^2-2x+1.$ 12. $2-x+2x^2.$

Art. 213; pages 178 and 179.

2.	31.	**8.**	2.02.	**14.**	.898.	**20.**	.3107.
3.	4.6.	**9.**	372.	**15.**	101.3.	**21.**	.7211.
4.	.88.	**10.**	21.6.	**16.**	.0534.	**22.**	1.077.
5.	123.	**11.**	.803.	**17.**	1.260.	**23.**	.6376.
6.	1.14.	**12.**	4.89.	**18.**	1.817.	**24.**	.8736.
7.	.098.	**13.**	.317.	**19.**	1.931.		

Art. 214; page 179.

1. $2x-3y$. 2. x^2-x+1. 3. x^2-2x-2.
4. a^2-1. 5. $2x+1$.

Art. 219; page 182.

21. 243. 23. 216. 25. -243. 27. 128.
22. 81. 24. 32. 26. 16. 28. 1296.

Art. 224; pages 184 and 185.

10. $4m^{\frac{9}{5}}$. 11. $6ac^{\frac{11}{15}}$. 14. $.5x^{-\frac{7}{3}}$. 15. $3a^{\frac{5}{2}}b^{\frac{5}{3}}$.
17. a^4-2+a^{-4}. 18. a^2-x^2. 19. $x^{-1}-1$.
20. $x^{-5}-3x^{-3}-4x^{-1}$. 21. $18a^2b^2+10+2a^{-2}b^{-2}$.
22. $6x^2-7x^{\frac{5}{3}}-19x^{\frac{4}{3}}+5x+9x^{\frac{2}{3}}-2x^{\frac{1}{3}}$.
23. $2x^{-1}y-10xy^{-1}+8x^3y^{-3}$. 24. $2-4a^{-\frac{4}{3}}x^{\frac{3}{2}}+2a^{-\frac{8}{3}}x^3$.
25. $18ab^{-2}-23+a^{-\frac{1}{2}}b+6a^{-1}b^2$.

Art. 225; pages 185 and 186.

5. $3c^{-\frac{9}{4}}$. 6. $m^{\frac{12}{5}}$. 7. $x^{\frac{13}{12}}$. 11. $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.
12. $a^{-\frac{4}{3}}-a^{-\frac{2}{3}}+1$. 13. $x^{-1}-9-10x$. 14. $x^{-\frac{3}{4}}+1$.
15. $m^{\frac{2}{3}}-2m^{\frac{1}{3}}n^{\frac{1}{3}}+n^{\frac{2}{3}}$. 17. $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.
16. $x^{-1}y^{-2}-x^{-2}y^{-3}-x^{-3}y^{-4}$. 18. $m^{-\frac{1}{3}}-n^{-1}+m^{\frac{1}{3}}n^{-2}$.

Art. 227; page 187.

10. y^3. 13. $c^{-\frac{1}{5}}$. 14. n^{-1}. 15. $x^{-\frac{1}{3}}$. 16. $a^{\frac{1}{m}}$.

Art. 229; pages 188 and 189.

8. $3x^{-2}-2x^{-1}-1$. 16. $x^{\frac{1}{6}}-3x^{\frac{1}{3}}+2x^{\frac{1}{2}}$. 21. a^{xy}.
9. $2x^{\frac{2}{3}}+x-4x^{\frac{4}{3}}$. 18. x^{3r-m}. 22. x.
10. $a^{\frac{3}{2}}b^{-\frac{1}{3}}-2+a^{-\frac{3}{2}}b^{\frac{1}{3}}$. 19. x^{-2ab}. 23. $\dfrac{x^{\frac{1}{2}}}{2a^{\frac{1}{2}}}$.
15. $2y^{-\frac{2}{3}}-y^{-\frac{1}{2}}$. 20. x^{a-b}.

24. $a^{\frac{3}{4}}b^{\frac{1}{4}} - a^{\frac{1}{4}}b^{\frac{3}{4}}$. 26. $\dfrac{7x^3}{(x^2+1)^{\frac{1}{3}}}$. 28. $\dfrac{(m+x)^{\frac{1}{2}}}{x^{\frac{1}{2}}}$.

25. $1+x$. 27. $(1-3x+x^2)^{-\frac{3}{2}}$. 29. $(1+x^2)^{\frac{1}{2}}$.

Art. 235; page 191.

9. $\sqrt[4]{5x^2}$. 10. $\sqrt{2a}$. 11. $\sqrt{7m^2n^3}$.
12. $\sqrt{5ab^3}$. 13. $\sqrt[m]{ab^2}$.

Art. 236; page 192.

12. $5xy\sqrt{xy^2-2x^2y}$. 15. $(x-3)\sqrt{a}$.
13. $3ab\sqrt[3]{2ab^2+5b}$. 16. $(2x+3)\sqrt{5}$.
14. $(x+y)\sqrt{x-y}$. 17. $(m-9n)\sqrt{3m}$.
19. $\frac{1}{2}\sqrt{6}$. 21. $\frac{1}{6}\sqrt{21}$. 23. $\frac{1}{2}\sqrt[3]{6x}$. 25. $\frac{6}{77}\sqrt{7}$.
20. $\frac{1}{6}\sqrt{30}$. 22. $\frac{2a}{9}\sqrt{3}$. 24. $\frac{1}{3}\sqrt[3]{15}$. 26. $\frac{3ab}{10cd}\sqrt{10bcd}$.
27. $\frac{y}{4a^3}\sqrt{14ax}$. 28. $\dfrac{b\sqrt{a^2+ax}}{2(a+x)}$. 29. $\dfrac{a\sqrt{abc}}{b^2(a+b)}$.

Art. 237; page 193.

10. $\sqrt{x^2-1}$. 11. $\sqrt{1-x^2}$. 12. $\sqrt{\dfrac{1+a}{1-a}}$. 13. $\sqrt{4-4x^2}$.

Art. 238; pages 194 and 195.

3. $5\sqrt{3}$. 9. $(2a+3b)\sqrt{b}$. 14. $\frac{3}{4}\sqrt[3]{2}+\frac{1}{3}\sqrt[3]{18}$.
4. $7\sqrt{6}$. 10. $9\sqrt{3}-7\sqrt{5}$. 15. $9\sqrt{3}$.
5. $3\sqrt{5}$. 11. $9\sqrt[3]{2}$. 16. $6a\sqrt{3a}$.
6. $\sqrt[3]{6}$. 12. $\frac{1}{5}\sqrt{5}$. 17. $(3a^2-2b^3)\sqrt{a+2b}$.
7. $20\sqrt{2}$. 18. $10\sqrt[3]{2}-2\sqrt[3]{3}$.
8. $\dfrac{\sqrt{15}}{15}$. 13. $\frac{7}{36}\sqrt{6}$. 19. $(a-4)\sqrt{7x}$.
20. $2\sqrt{x^2-y^2}$.

Art. 239; pages 195 and 196.

2. $\sqrt[6]{4}$, $\sqrt[6]{27}$.
3. $\sqrt[6]{125}$, $\sqrt[6]{16}$, $\sqrt[6]{9}$.
4. $\sqrt[12]{625}$, $\sqrt[12]{216}$, $\sqrt[12]{49}$.
5. $\sqrt[15]{32a^5}$, $\sqrt[15]{27b^3}$, $\sqrt[15]{64c^3}$.
6. $\sqrt[24]{x^4y^4}$, $\sqrt[24]{y^6z^6}$, $\sqrt[24]{z^3x^3}$.
7. $\sqrt[12]{a^2+2ab+b^2}$, $\sqrt[12]{a^3-3a^2b+3ab^2-b^3}$.
8. $\sqrt[3]{2}$. 9. $\sqrt[5]{5}$. 10. $\sqrt{3}$, $\sqrt[4]{7}$, $\sqrt[3]{4}$.

Art. 240; pages 196 to 198.

3. $6\sqrt{7}$.
4. $75\sqrt{6}$.
5. $30x\sqrt{5}$.
6. $a\sqrt[3]{b^2c^2}$.
7. $\sqrt[3]{3}$.
8. $\sqrt[12]{432}$.
9. $\sqrt[6]{a^3b^2x^5}$.
10. $2a\sqrt[6]{2a}$.
11. $12\sqrt[10]{288}$.
12. $\sqrt{xyz}$.
13. $\sqrt[6]{3}$.
14. $\sqrt[15]{72x^2}$.
17. $x+\sqrt{x}-6$.
18. $4-5\sqrt{10}$.
19. $2x-7\sqrt{3x}-12$.
20. $8a-10\sqrt{ab}-3b$.
21. $x-y+2\sqrt{yz}-z$.
22. $2-3\sqrt{x^2+x}$.
23. $4+2\sqrt{10}$.
24. $28+8\sqrt{42}$.
25. $11-20\sqrt{15}+5\sqrt{10}$.
26. $21-12\sqrt{3}$.
27. $147+60\sqrt{6}$.
28. $1+2a\sqrt{1-a^2}$.
29. $2a-2\sqrt{a^2-b^2}$. 30. 1. 31. 2. 32. $6x+49$.

Art. 241; page 199.

2. $3\sqrt{2}$.
5. $\sqrt[6]{24}$.
6. $\sqrt[6]{\frac{3}{2}}$.
7. $\sqrt[20]{\frac{16}{243}}$.
8. $\sqrt[15]{54}$.
9. $\sqrt[12]{32a}$.
10. $\sqrt[15]{\frac{9a}{8b}}$.
11. $\sqrt[12]{18yz}$.

Art. 242; pages 199 and 200.

5. $a^3x\sqrt[3]{ax^2}$.
6. $3\sqrt{2}$.
8. $192x\sqrt{3x}$.
9. $2\sqrt[4]{2}$.
10. $81a^4bx\sqrt[3]{bx}$.
11. $18ab\sqrt[3]{3ab^2}$.
17. $\sqrt[5]{3a}$. 19. $\sqrt[3]{x-1}$. 20. $\sqrt{3}$. 21. $\sqrt[5]{x^2y^3}$. 22. $\sqrt{2}$.

Art. 243; page 201.

4. $\frac{2}{5y}\sqrt{5xy}$. 5. $\frac{5}{3a}\sqrt[3]{3a}$. 6. $\frac{1}{2x}\sqrt[5]{2x^2}$. 7. $\frac{2c}{3a}\sqrt[4]{3a^2}$.

Art. 244; page 202.

3. $\frac{12-4\sqrt{2}}{7}$.
4. $5+2\sqrt{3}$.
5. $2\sqrt{6}-5$.
6. $\frac{a+2\sqrt{ab}+b}{a-b}$.
7. $-\frac{16+7\sqrt{10}}{13}$.
8. $\frac{a^2-2a\sqrt{x}+x}{a^2-x}$.
9. $\frac{a-1-\sqrt{a+1}}{a}$.
10. $x+1-\sqrt{x^2+2x}$.
11. $2a^2-1-2a\sqrt{a^2-1}$.
12. $\frac{x^2-2+x\sqrt{x^2-4}}{2}$.
13. $\frac{a+\sqrt{a^2-x^2}}{x}$.
14. $\sqrt{a^4-1}-a^2$.

Art. 245; page 203.

2. 7.243. 3. 3.365. 4. .101. 5. .269.

Art. 249; page 205.

3. $-8\sqrt{6}$. 5. $12\sqrt{ab}$. 7. $5-5\sqrt{-1}$. 9. 12.
4. $-ax$. 6. $-2\sqrt{-15}$. 8. 46. 10. 6Q.
11. $1-4\sqrt{-3}$. 14. a^2+b. 19. $2\sqrt{2}$.
12. $-11-4\sqrt{6}$. 15. y^3-x^3. 20. $\sqrt{2}$.
13. 2. 16. 0. 21. $\sqrt{3}$.

Art. 256; page 208.

4. $\sqrt{5}-2$. 7. $2\sqrt{2}+\sqrt{7}$. 10. $3+\sqrt{5}$.
5. $3+\sqrt{7}$. 8. $3-\sqrt{3}$. 11. $3\sqrt{2}+\sqrt{5}$.
6. $\sqrt{5}-\sqrt{3}$. 9. $\sqrt{15}-\sqrt{5}$. 13. $5+2\sqrt{2}$.
14. $3\sqrt{5}-\sqrt{2}$. 16. $\sqrt{m+n}-\sqrt{m-n}$.
15. $7-3\sqrt{2}$. 17. $\sqrt{a+x}+\sqrt{a}$.

Art. 257; pages 209 and 210.

3. 2. **8.** $\frac{2}{3}$. **13.** -3. **18.** -3. **23.** $\frac{13}{4}$.

4. 4. **9.** 4. **14.** $2a$. **19.** 25. **24.** $\frac{ab}{a+b}$.

5. 6. **10.** 81. **15.** -1. **20.** 12. **25.** $-b$.

6. 5. **11.** 4. **16.** 4. **21.** 3. **26.** 3.

7. -2. **12.** 8. **17.** 5. **22.** $\frac{a}{3}$. **27.** $3a-1$.

Art. 259; page 212.

3. ± 3. **7.** $\pm\sqrt{\frac{c-b}{a}}$. **11.** ± 3. **15.** $\pm\frac{a-b}{2}$.

4. ± 5. **8.** ± 1. **12.** ± 2. **16.** ± 2.

5. $\pm\frac{2}{3}$. **9.** $\pm\sqrt{19}$. **13.** $\pm\sqrt{a+b}$. **17.** ± 1.

6. $\pm 2\sqrt{-1}$. **10.** ± 2. **14.** $\pm\sqrt{2}$. **18.** $\pm\frac{1}{2}$.

Art. 262; page 215.

3. $1, -5$. **5.** $4, 3$. **7.** $2, -\frac{2}{3}$. **9.** $\frac{3}{2}, \frac{1}{2}$. **11.** $1, -\frac{3}{2}$.

4. $4, 1$. **6.** $2, -3$. **8.** $-2, -\frac{1}{2}$. **10.** $3, -\frac{1}{4}$. **12.** $\frac{7}{3}, -\frac{2}{3}$.

Art. 263; page 216.

2. $-2, \frac{5}{4}$. **4.** $\frac{2}{5}, \frac{1}{5}$. **6.** $-1, -\frac{13}{2}$. **8.** $\frac{3}{4}, -\frac{4}{9}$.

3. $1, -\frac{11}{9}$. **5.** $1, \frac{3}{4}$. **7.** $2, -\frac{17}{8}$. **9.** $\frac{1}{2}, \frac{1}{3}$.

10. $\frac{1}{4}, -\frac{7}{8}$. **11.** $\frac{3}{4}, -\frac{1}{12}$.

Art. 265; page 218.

3. $-3, \frac{1}{2}$. 6. $-2, \frac{2}{3}$. 9. $2, \frac{2}{7}$. 12. $\frac{2}{3}, -\frac{1}{2}$.

4. $1, -\frac{3}{4}$. 7. $\frac{1}{2}, -\frac{3}{4}$. 10. $-4, -\frac{5}{3}$. 13. $-\frac{1}{2}, -\frac{1}{4}$.

5. $6, -3$. 8. $5, -\frac{3}{2}$. 11. $3, -\frac{1}{5}$. 14. $\frac{3}{2}, -\frac{1}{3}$.

15. $\frac{1}{3}, \frac{1}{5}$. 16. $\frac{7}{3}, \frac{2}{5}$.

Art. 266; pages 218 to 221.

1. $-\frac{1}{2}, -\frac{1}{6}$.
2. $-1, -4$.
3. $1, -\frac{7}{4}$.
4. $\frac{5}{2}, \frac{15}{4}$.
5. $13, -2$.
6. $\frac{1}{2}, \frac{1}{14}$.
7. $1, \frac{13}{4}$.
8. $-\frac{3}{2}, -\frac{1}{2}$.
9. $-4, -\frac{11}{6}$.
10. $5, -3$.
11. $4 \pm 2\sqrt{3}$.
12. $4, -\frac{7}{4}$.
13. $-10 \pm \sqrt{78}$.
14. $1, \frac{7}{36}$.
15. $2, \frac{1}{3}$.
16. $4, -\frac{5}{3}$.
17. $-\frac{7}{2}, -\frac{5}{2}$.
18. $5, \frac{16}{5}$.
19. $4, \frac{1}{4}$.
20. $1, \frac{2}{9}$.
21. $-2, \frac{16}{23}$.
22. $1, -18$.
23. $-3, \frac{51}{22}$.
24. 2.
25. $4, 0$.
26. $-2, \frac{12}{65}$.
27. $3, \frac{7}{9}$.
28. $4, -\frac{16}{3}$.
29. $3, -2$.
30. $5, -\frac{1}{2}$.
33. $a+b, a-b$.
34. $a, -b$.
35. $1, a$.
36. $-16, -2c$.
37. $m^2, -m^3$.
38. $\frac{b}{a}, \frac{d}{c}$.

39. $2p,\ -5p.$ **41.** $\frac{3a}{4}, \frac{a}{2}.$ **43.** $\frac{a}{b}, \frac{b}{a}.$

40. $-\frac{3a}{2}, -\frac{b}{3}.$ **42.** $m,\ -\frac{m}{m+1}.$ **44.** $a, \frac{1}{a}.$

45. $a+b,\ 4b.$

46. $(a+b)^2,\ -(a-b)^2.$

47. $9m,\ -m.$

48. $a,\ -2a.$

49. $-a,\ -2a.$

50. $\frac{a+b}{c}, \frac{c}{a+b}.$

51. $-a,\ -b.$

52. $a-b,\ -a-c.$

53. $\frac{2a-b}{ac},\ -\frac{3a+2b}{bc}.$

54. $\frac{a+b}{a-b}, \frac{a-b}{a+b}.$

Art. 267; page 222.

3. $2,\ -\frac{9}{2}.$ **5.** $5,\ 2.$ **7.** $-1,\ -\frac{1}{6}.$

4. $\frac{5}{3},\ -1.$ **6.** $-2,\ \frac{9}{5}.$ **8.** $2,\ \frac{1}{5}.$

9. $\frac{5}{2},\ -\frac{1}{2}.$ **11.** $\frac{8}{3},\ \frac{2}{3}.$

10. $-\frac{7}{2},\ -\frac{2}{3}.$ **12.** $\frac{9}{2},\ -\frac{3}{5}.$

Art. 268; pages 224 to 227.

3. 10 barrels, at $17.50. **5.** 21, 6.

4. 9, 6. **6.** Length, 125; breadth, 50.

7. 14, 5. **12.** 16. **17.** 9.

8. 7, 8. **13.** 16 barrels, at $6. **18.** 15, 9.

9. 16, 10. **14.** 13, 6. **19.** 3712.

10. 5, 3. **15.** 3 inches. **20.** $80 or $20.

11. 7, 8, 9. **16.** $30. **21.** 20.

22. Area of court, 529 sq. yds.; width of walk, 4 yds.

23. 36 bushels, at $1.40. **24.** 84.

25. Larger pipe, 5 hours; smaller, 7 hours.

26. $2000. **27.** 5. **28.** 6. **29.** 343 cubic inches.

30. First, 14,400; second, 625; or, first, 8464; second, 6561.

31. 38 or 266 miles. **32.** 70 miles.

33. 100, at $15 each.

Art. 270; pages 229 and 230.

4. $\pm 3, \ \pm 4$. **7.** $\pm\frac{1}{2}, \ \pm\frac{1}{5}\sqrt{5}$. **10.** $\pm 1, \ \pm 2$.

5. $\sqrt[3]{3}, \ -\sqrt[3]{23}$. **8.** $\pm 1, \ \pm\frac{1}{9}$. **11.** $2, \ -3$.

6. $1, \ -2$. **9.** $2, \ -\frac{3}{2}$. **12.** $4, \ \sqrt[3]{49}$.

13. $243, \ -28\sqrt[3]{784}$. **14.** $(\pm 2)^n, \ \left(-\frac{14}{3}\right)^{\frac{n}{2}}$.

15. $64, \ \left(\frac{97}{3}\right)^{\frac{6}{5}}$. **17.** $\frac{1}{4}, \ \left(\frac{2}{3}\right)^{\frac{2}{5}}$. **19.** $4, \ \frac{64}{9}$.

16. $16, \ \left(\frac{2}{3}\right)^{\frac{4}{3}}$. **18.** $25, \ \frac{9}{16}$. **20.** $\pm 8, \ \left(-\frac{74}{5}\right)^{\frac{3}{4}}$.

21. $-32, \ 2^{-\frac{5}{3}}$.

Art. 271; page 232.

4. $2, \ -2, \ 3, \ 7$. **11.** $8, \ -2, \ 3 \pm \sqrt{110}$.

5. $2, \ -2, \ -3, \ -7$. **12.** $6, \ -9$.

6. $1, \ 9, \ 5 \pm 2\sqrt{2}$. **13.** $\frac{3}{2}, \ -\frac{9}{2}, \ \frac{-3 \pm 2\sqrt{3}}{2}$.

7. $\pm 2, \ \pm\sqrt{11}$. **14.** $-2, \ -\sqrt[3]{15}$.

8. $3, \ -\frac{9}{2}, \ \frac{-3 \pm \sqrt{-55}}{4}$. **15.** $0, \ 2, \ \frac{7}{8}, \ \frac{9}{8}$.

9. $2, \ -3, \ 4, \ -5$. **16.** $1, \ -1, \ -6, \ -8$.

10. $1, \ 2, \ -5, \ 8$. **17.** $0, \ -5, \ \frac{1}{3}, \ -\frac{16}{3}$.

18. $a + b^{\frac{3}{5}}, \ a + 3\sqrt[5]{3b^3}$.

Art. 274; page 234.

Note. In this, and the three following articles, the answers are arranged in the order in which they are to be taken; thus, in Ex. 2, the value $x=1$ is to be taken with $y=-2$, and $x=-\frac{7}{25}$ with $y=\frac{46}{25}$.

2. $x=1,\ -\frac{7}{25}$; $y=-2,\ \frac{46}{25}$.

3. $x=7,\ -8$; $y=-8,\ 7$.

4. $x=9,\ -6$; $y=6,\ -9$.

5. $x=2,\ -\frac{1}{3}$; $y=4,\ \frac{5}{3}$.

6. $x=3,\ -4$; $y=4,\ -3$.

7. $x=2,\ 3$; $y=-3,\ -2$.

8. $x=5,\ -2$; $y=-2,\ 5$.

9. $x=4,\ y=6$.

10. $x=6,\ -4$; $y=-4,\ 6$.

11. $x=3,\ -\frac{15}{13}$; $y=2,\ \frac{62}{13}$.

12. $x=\frac{1}{2},\ -1$; $y=2,\ -1$.

13. $x=3,\ \frac{47}{3}$; $y=2,\ -\frac{13}{3}$.

14. $x=4,\ -2$; $y=8,\ -1$.

Art. 275; page 237.

4. $x=3,\ -2$; $y=-2,\ 3$.

5. $x=9,\ -3$; $y=3,\ -9$.

6. $x=-3,\ -7$; $y=7,\ 3$.

7. $x=2,\ -3$; $y=-3,\ 2$.

8. $x=\pm 4,\ \pm 3$; $y=\pm 3,\ \pm 4$.

9. $x=3,\ -7$; $y=-7,\ 3$.

10. $x=5,\ -3$; $y=3,\ -5$.

11. $x=2,\ 1$; $y=1,\ 2$.

12. $x=8,\ -\frac{15}{2}$; $y=\frac{15}{2},\ -8$.

13. $x=\pm 7,\ \pm 6$; $y=\pm 6,\ \pm 7$.

14. $x=3,\ -7$; $y=7,\ -3$.

15. $x=7,\ -12$; $y=-12,\ 7$.

16. $x=15,\ -3$; $y=-3,\ 15$.

17. $x=-1,\ -4$; $y=4,\ 1$.

18. $x=6,\ -11$; $y=11,\ -6$.

19. $x=7,\ -9$; $y=-9,\ 7$.

Art. 276; page 239.

2. $x = \pm 7,\ \pm \frac{5}{2}\sqrt{2}$;
 $y = \pm 2,\ \mp \frac{9}{2}\sqrt{2}$.

3. $x = \pm 3,\ \pm \frac{5}{3}\sqrt{3}$;
 $y = \mp 1,\ \mp \frac{1}{3}\sqrt{3}$.

4. $x = \pm 2,\ \pm \frac{1}{5}\sqrt{5}$;
 $y = \mp 3,\ \pm \frac{8}{5}\sqrt{5}$.

5. $x = \pm 2,\ \pm \frac{1}{5}\sqrt{10}$;
 $y = \pm \frac{1}{2},\ \mp \frac{2}{5}\sqrt{10}$.

6. $x = \pm 3,\ \pm 2\sqrt{2}$;
 $y = \pm 1,\ \pm \sqrt{2}$.

7. $x = \pm 3,\ \pm \frac{5}{3}$;
 $y = \pm 5,\ \pm \frac{13}{3}$.

8. $x = \pm 2,\ \pm 7$;
 $y = \pm 1,\ \mp 19$.

9. $x = \pm 2,\ \pm \frac{5}{31}\sqrt{31}$;
 $y = \pm 3,\ \mp \frac{6}{31}\sqrt{31}$.

10. $x = \pm 2,\ \pm \frac{5}{11}\sqrt{11}$;
 $y = \mp 1,\ \pm \frac{7}{11}\sqrt{11}$.

11. $x = \pm 1,\ \pm 2$;
 $y = \pm \frac{3}{2},\ \pm \frac{9}{8}$.

Art. 277; pages 242 and 243.

5. $x = -5,\ -\frac{3}{2}$;
 $y = 1,\ -\frac{4}{3}$.

6. $x = 1,\ 8$;
 $y = 8,\ 1$.

7. $x = 4,\ -4$;
 $y = \pm 5,\ \pm 5$.

8. $x = 2,\ 3$;
 $y = 3,\ 2$.

9. $x = -2,\ \frac{20}{11}$;
 $y = -5,\ \frac{8}{11}$.

10. $x = 2m,\ -m$;
 $y = m,\ -2m$.

11. $x = 3,\ 2,\ -3 \pm \sqrt{3}$;
 $y = 2,\ 3,\ -3 \mp \sqrt{3}$.

12. $x = \pm 2,\ \pm \frac{8}{5}\sqrt{5}$;
 $y = \mp 3,\ \mp \frac{1}{5}\sqrt{5}$.

13. $x = 8,\ 4$;
 $y = 4,\ 8$.

14. $x = \pm 2,\ \pm 14$;
 $y = \mp 3,\ \mp 5$.

15. $x = \frac{1}{2}, \frac{1}{9};$
$y = \frac{1}{9}, \frac{1}{2}.$

16. $x = 2a, -a-b;$
$y = a+b, -2a.$

17. $x = \frac{5}{2}, -3;$
$y = \pm\frac{1}{2}, \pm\sqrt{3}.$

18. $x = \frac{1}{4}, \frac{1}{7};$
$y = -\frac{1}{7}, -\frac{1}{4}.$

19. $x = 3, 4, -4 \pm \sqrt{-11};$
$y = 4, 3, -4 \mp \sqrt{-11}.$

20. $x = \pm 3, \pm 2;$
$y = \mp 2, \mp 3.$

21. $x = 1, -3, 1 \pm \sqrt{-2};\ y = -3, 1, 1 \mp \sqrt{-2}.$

22. $x = 1, 2, \frac{3 \pm \sqrt{-55}}{2};\ y = -2, -1, \frac{-3 \pm \sqrt{-55}}{2}.$

23. $x = 2a, -a;$
$y = a, -2a.$

24. $x = 4, 9;$
$y = 9, 4.$

25. $x = a-b, b-a;$
$y = a+b, 2a.$

26. $x = 2, 3;$
$y = 3, 2.$

27. $x = b \pm a;$
$y = -b \pm a.$

28. $x = 4, 16, -12 \pm \sqrt{58};$
$y = 5, -7, -1 \mp \sqrt{58}.$

29. $x = 9, 5\frac{20}{117};$
$y = 4, \frac{20}{117}.$

30. $x = \pm 2, \pm\frac{1}{7}\sqrt{7};$
$y = \pm 5, \mp\frac{11}{7}\sqrt{7}.$

31. $x = 2a-b, a-2b;$
$y = a-2b, 2a-b.$

32. $x = 2, 1, \frac{3 \pm \sqrt{-19}}{2};$
$y = 1, 2, \frac{3 \mp \sqrt{-19}}{2}.$

33. $x = 2, -\frac{1}{4};$
$y = -1, 2.$

34. $x = 4, \frac{4}{9};$
$y = 2, \frac{22}{3};$
$z = 3, \frac{59}{9}.$

35. $x = \pm 1, \pm\frac{59}{4};$
$y = \mp 3, \pm\frac{31}{8}.$

36. $x = 3, -7;$
$y = -1, -21.$

37. $x = 3, 2, \frac{-9 \pm \sqrt{309}}{12};$
$y = 2, 3, \frac{-9 \mp \sqrt{309}}{12}.$

Art. 278; pages 243 to 245.

1. 9, 5.
2. 13, 6.
3. 900 square rods, 400 square rods.
4. 3, 4.
5. Length, 10 rods; breadth, 6 rods.
6. 7, 4.
7. Duck, $1.75; turkey, $2.25.
8. 21 or 12.
9. $\frac{3}{5}$ or $\frac{-16}{24}$.
10. Length, 150 feet; breadth, 100 feet.
11. Length, 16 rods; breadth, 10 rods; or, length, $13\frac{1}{3}$ rods; breadth, 12 rods.
12. Rate of the boatman, 4 miles an hour; of the stream, 2 miles an hour.
13. A, 40 acres at $8; B, 64 acres at $5.
14. 7, 5.
15. 5, 2.
16. Hind-wheel, 4 yards; fore-wheel, 3 yards.
17. First rate, 7 per cent; second, 6 per cent.
18. Length, 16 yards; width, 2 yards.

Art. 281; page 247.

9. $x^2-9x=-20.$
10. $x^2+2x=3.$
11. $5x^2-12x=9.$
12. $3x^2-2x=133.$
13. $12x^2-17x=-6.$
14. $21x^2+44x=32.$
15. $6x^2+31x=-35.$
16. $3x^2+17x=0.$
17. $x^2-2ax-bx=-a^2-ab+2b^2.$
18. $x^2-2mx=m^4-m^2.$
19. $x^2-4x=-1.$
20. $4x^2-4mx=n-m^2.$

Art. 282; page 249.

6. $0, \frac{1}{2}.$
9. $2, -4, -5.$
10. $0, \pm 3.$
12. $0, -4.$
13. $\pm a, \frac{a \pm \sqrt{a^2+4b}}{2}.$
14. $0, \frac{3}{4}, -\frac{2}{3}.$
16. $-1, 2, \pm 3, \pm 4.$
17. $-1, \frac{1 \pm \sqrt{-3}}{2}.$
18. $\frac{3}{2}, \frac{-3 \pm 3\sqrt{-3}}{4}.$
19. $\pm 1, \frac{\pm 1 \pm \sqrt{-3}}{2}.$

20. $-1, \pm\sqrt{-1}$.
23. $a, \pm a\sqrt{-1}$.
21. $1, \pm 3$.
22. $-\frac{3}{2}, \pm 1$.
24. $-\frac{2}{3}, \pm\frac{3}{2}$.

Art. 283; pages 251 to 253.

5. $(x+6)(x-10)$.
6. $(x+8)(x+5)$.
7. $(x-2)(x-9)$.
8. $(2x+3)(x-5)$.
9. $(4x-3)(x-3)$.
10. $(x+7)(5x+1)$.
11. $(13+x)(3-x)$.
12. $(1+2x)(2-3x)$.
13. $(x+2+\sqrt{3})(x+2-\sqrt{3})$.
14. $(3x-1+\sqrt{5})(3x-1-\sqrt{5})$.
15. $(4x-3)(2x-3)$.
16. $(2+x)(3-2x)$.
17. $(1+2x)(5-6x)$.
18. $(3x-2+\sqrt{3})(3x-2-\sqrt{3})$.
19. $(5+2x)(1-4x)$.
20. $(10x-3)(x-2)$.
21. $(2x+3)(8x+5)$.
22. $(\sqrt{17}+4+x)(\sqrt{17}-4-x)$.
23. $(5+12x)(3-2x)$.
24. $(5x-2+\sqrt{6})(5x-2-\sqrt{6})$.
25. $(6x+5a)(x-3a)$.
27. $(4x+5y)(3x-2y)$.
26. $(4x+5m)(5x+4m)$.
28. $(7x-3mn)(3x-7mn)$.
29. $(x+3y-2)(x-2y+3)$.
30. $(x+2y+2)(x+y+1)$.
31. $(3+2y-x)(2-3y+x)$.
32. $(x-2y-4z)(x-3y-z)$.
33. $(2x+y-1)(x-y+2)$.
34. $(a+b+3)(3a+b-4)$.
38. $(x^2+x+1)(x^2-x+1)$.
39. $(x^2+3x+1)(x^2-3x+1)$.
40. $(2a^2+2ab-b^2)(2a^2-2ab-b^2)$.
41. $(m^2+4mn+n^2)(m^2-4mn+n^2)$.
42. $(1+3b-2b^2)(1-3b-2b^2)$.
43. $(x^2+4xy+2y^2)(x^2-4xy+2y^2)$.

44. $(2a^2+2a+3)(2a^2-2a+3)$.

45. $(2m^2+2m-5)(2m^2-2m-5)$.

46. $(a^2+ax\sqrt{3}-x^2)(a^2-ax\sqrt{3}-x^2)$.

47. $(x^2+3x\sqrt{2}+9)(x^2-3x\sqrt{2}+9)$.

48. $(2a^2+ab+4b^2)(2a^2-ab+4b^2)$.

49. $(4x^2+5mx-3m^2)(4x^2-5mx-3m^2)$.

50. $(3x^2+3x\sqrt{2}+2)(3x^2-3x\sqrt{2}+2)$.

51. $(3a^2+4am+5m^2)(3a^2-4am+5m^2)$.

52. $(2+2n-7n^2)(2-2n-7n^2)$.

53. $(4x^2+3xy-5y^2)(4x^2-3xy-5y^2)$.

Art. 284; page 253.

1. $\pm\sqrt{2}\pm\sqrt{-2}$.

2. $\pm 1\pm\sqrt{2}$.

3. $\dfrac{\pm\sqrt{3}\pm\sqrt{-1}}{2}$.

4. $\dfrac{a}{2}\left(\pm\sqrt{2}\pm\sqrt{-2}\right)$.

5. $\pm 1\pm\sqrt{3}$.

6. $\dfrac{\pm\sqrt{14}\pm\sqrt{-2}}{4}$.

Art. 295; pages 259, 260.

3. $x<4$.

4. $x<1$.

5. $x>1\frac{6}{7}$.

6. $x<2a$.

7. $x<a+b$.

8. $x>2, \quad y>4$.

9. $x<24, \quad y>3$.

10. $x>5$ and <15.

11. 8.

12. 19.

13. 32 or 33.

Art. 325; pages 274, 275.

3. 4.

4. 11.

5. $\dfrac{9}{35}$.

6. $\dfrac{25}{27}$.

7. 12.

8. $14\sqrt{3}$.

9. $\dfrac{5}{9}\sqrt{3}$.

10. $3, -\dfrac{1}{11}$.

11. $2, -3, \dfrac{7}{9}$.

12. $\pm\dfrac{a}{b}$.

13. $x=\pm 6$,
$y=\pm 10$.

14. $x=\pm a^2b$,
$y=\pm ab^2$.

15. 25, 20.

16. 23, 27.

17. 9, 3.

18. 8, 18.

19. 26, 14.

24. 17, 12.

25. 12, 8. **26.** A, \$105; B, \$189; C, \$270. **27.** 8 : 7.

28. First, 1 : 2; second, 2 : 1. **29.** $\frac{a+b}{2}$, $\frac{a-b}{2}$.

Art. 335; pages 278, 279.

3. 63.
4. $y=\frac{5}{3}x^2$.
5. 6.
6. 70.
7. 14.
8. $\frac{21}{10}$.
9. $\pm\frac{1}{3}\sqrt{3}$.
10. $5x$ and $-\frac{6}{x}$.
11. 9.
12. $3(\sqrt{2}-1)$ inches.
13. 10 inches.
14. 6 inches.
15. 10.
16. 8.
17. $y=\frac{14}{4-5x}$.

Art. 340; page 281.

2. $l=71$, $S=540$.
3. $l=-69$, $S=-620$.
4. $l=57$, $S=552$.
5. $l=-145$, $S=-2175$.
6. $l=\frac{35}{4}$, $S=\frac{315}{2}$.
7. $l=-\frac{3}{5}$, $S=0$.
8. $l=\frac{23}{12}$, $S=\frac{62}{3}$.
9. $l=-\frac{5}{11}$, $S=\frac{1}{2}$.
10. $l=5$, $S=17$.
11. $l=\frac{137}{15}$, $S=\frac{917}{15}$.

Art. 341; pages 283 to 285.

4. $a=3$, $S=741$.
5. $a=1\frac{1}{2}$, $l=-12\frac{1}{2}$.
6. $d=\frac{1}{3}$, $S=39$.
7. $d=-\frac{1}{12}$, $l=-1\frac{1}{4}$.
8. $d=-2\frac{1}{2}$, $n=13$.
9. $a=5$, $d=-3$.
10. $n=16$, $l=-43$.
11. $n=18$, $S=411$.

12. $a=3$, $l=-61$.

13. $a=\frac{1}{2}, -\frac{1}{6}$; $n=10, 12$.

14. $a=-\frac{1}{2}$, $d=2$.

15. $d=-\frac{1}{15}$, $n=16$.

16. $d=\frac{5}{4}$, $l=\frac{67}{4}$.

17. $a=4, -5$; $n=52, 43$.

18. $n=14$, $l=-15\frac{2}{3}$.

20. $d=\frac{l-a}{n-1}$.

21. $d=\frac{2(S-an)}{n(n-1)}$, $l=\frac{2S-an}{n}$.

22. $a=\frac{2S-n(n-1)d}{2n}$, $l=\frac{2S+n(n-1)d}{2n}$.

23. $n=\frac{l-a+d}{d}$, $S=\frac{(l+a)(l-a+d)}{2d}$.

24. $a=l-(n-1)d$, $S=\frac{n}{2}[2l-(n-1)d]$.

25. $a=\frac{2S-nl}{n}$, $d=\frac{2(nl-S)}{n(n-1)}$.

26. $l=\frac{-d\pm\sqrt{8dS+(2a-d)^2}}{2}$.

27. $d=\frac{l^2-a^2}{2S-a-l}$, $n=\frac{2S}{a+l}$.

28. $n=\frac{2l+d\pm\sqrt{(2l+d)^2-8dS}}{2d}$, $a=\frac{d\mp\sqrt{(2l+d)^2-8dS}}{2}$.

Art. 344; page 286.

1. $d=\frac{1}{3}$.
2. $d=-\frac{1}{2}$.
3. $d=-1\frac{1}{5}$.
4. $d=\frac{4}{7}$.
5. $d=-\frac{1}{5}$.
6. $\frac{4}{15}$.
7. $2ab$.
8. $\frac{a^2+b^2}{a^2-b^2}$.

Art. 345; pages 287, 288.

3. 2500.
4. -43.
5. 4, 11, 18, 25.
6. 11.
7. $l=10m-27n$, $S=55m-135n$.
8. $\frac{23a}{3}$.
9. 62750.
10. 2, 6, 10, 14; or, -2, -6, -10, -14.
11. 22.

12. $-4, -1, 2, 5, 8$; or, $\frac{59}{14}, \frac{16}{7}, \frac{5}{14}, -\frac{11}{7}, -\frac{7}{2}$.

13. After 9 days, at a distance of 90 leagues.

14. $4117\frac{1}{8}$ feet.

15. 3, 7, 11; or, $4\frac{1}{2}, 7\frac{1}{2}, 10\frac{1}{2}$.

16. 8.

17. $2950.

18. 852.

Art. 349; pages 290, 291.

3. $l=256$, $S=511$.

4. $l=\frac{64}{243}$, $S=\frac{2059}{243}$.

5. $l=2048$, $S=1638$.

6. $l=-\frac{1}{256}$, $S=\frac{341}{256}$.

7. $l=\frac{1}{2048}$, $S=\frac{2047}{2048}$.

8. $l=-\frac{729}{64}$, $S=-\frac{1261}{192}$.

9. $l=\frac{1}{32}$, $S=\frac{511}{32}$.

10. $l=-\frac{1}{324}$, $S=\frac{91}{162}$.

11. $l=192$, $S=129$.

12. $l=-\frac{1}{768}$, $S=-\frac{341}{256}$.

Art. 350; pages 292, 293.

3. $a=\frac{1}{2}$, $S=\frac{1023}{2}$.

4. $a=-\frac{3}{2}$, $l=48$.

5. $r=3, -3$; $S=2186, 1094$.

6. $n=5$, $S=121$.

7. $a=\frac{2}{3}$, $l=\frac{2}{6561}$.

8. $r=-\frac{1}{4}$, $S=\frac{2457}{1024}$.

9. $r=\frac{1}{2}$, $n=7$.

10. $l=-\frac{243}{2}$, $n=6$.

11. $a=-1$, $n=8$.

12. $l=\frac{a+(r-1)S}{r}$.

13. $r=\frac{S-a}{S-l}$.

14. $a=rl-(r-1)S$.

15. $a=\frac{l}{r^{n-1}}$, $S=\frac{l(r^n-1)}{r^{n-1}(r-1)}$.

16. $a=\frac{(r-1)S}{r^n-1}$, $l=\frac{r^{n-1}(r-1)S}{r^n-1}$.

17. $r=\sqrt[n-1]{\frac{l}{a}}$, $S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$.

Art. 351; page 294.

2. 4. 4. $-\frac{3}{4}$. 6. $\frac{9}{4}$. 8. $-\frac{160}{19}$.

3. $\frac{8}{3}$. 5. $-\frac{15}{4}$. 7. $\frac{30}{11}$. 9. $\frac{x^2}{a^2+x^2}$.

Art. 352; page 295.

2. $\frac{8}{11}$. 3. $\frac{11}{27}$. 4. $\frac{11}{15}$. 5. $\frac{86}{165}$. 6. $\frac{91}{825}$. 7. $\frac{237}{1100}$.

Art. 355; pages 295, 296.

1. 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$, $\frac{32}{81}$, $\frac{64}{243}$.

2. $\pm\frac{3}{2}$, $\frac{9}{2}$, $\pm\frac{27}{2}$, $\frac{81}{2}$, $\pm\frac{243}{2}$.

3. -6, -18, -54, -162, -486, -1458.

4. $\pm\frac{3}{4}$, $\frac{3}{8}$, $\pm\frac{3}{16}$, $\frac{3}{32}$, $\pm\frac{3}{64}$, $\frac{3}{128}$, $\pm\frac{3}{256}$.

5. ± 4, -8, ± 16, -32, ± 64.

6. $-\frac{9}{4}$, $\frac{27}{16}$, $-\frac{81}{64}$, $\frac{243}{256}$. 7. 5. 8. $4x^2-9y^2$. 9. $\frac{a}{b}$.

Art. 356; pages 296, 297.

2. 3. 3. 5, 10, 20, 40; or, -15, 30, -60, 120.

4. 5, 15, 45; or, 40, -20, 10. 5. ± 4. 6. \$64.

7. 3100 feet. 8. 2, 4, 8, 16; or, $\frac{810}{13}$, $-\frac{540}{13}$, $\frac{360}{13}$, $-\frac{240}{13}$.

9. $\frac{81}{8192}$. 10. 3, 9, 27. 12. 1, 2, 4.

11. 2, 4, 8; or, -2, 4, -8.

Art. 361; page 300.

3. $\frac{3}{74}$. **4.** $-\frac{1}{78}$. **5.** $-\frac{5}{4}$. **6.** $\frac{3}{4}$. **7.** $-\frac{3}{142}$.

8. $\frac{48}{125}, \frac{24}{65}, \frac{16}{45}, \frac{12}{35}, \frac{48}{145}, \frac{8}{25}, \frac{48}{155}$. **11.** 15.

9. $-\frac{40}{17}, -\frac{20}{7}, -\frac{40}{11}, -5$. **12.** $\frac{a^2-b^2}{a^2+b^2}$.

10. $7, -21, -\frac{21}{5}, -\frac{7}{3}, -\frac{21}{13}, -\frac{21}{17}$.

13. $\frac{ab}{na-nb-a+2b}$. **14.** $\frac{ab(m+1)}{bm+2a-b}$. **15.** $-\frac{3}{19}$.

Art. 365; pages 303, 304.

2. $c^{\frac{8}{3}}+4c^2d^{-\frac{3}{4}}+6c^{\frac{4}{3}}d^{-\frac{3}{2}}+4c^{\frac{2}{3}}d^{-\frac{9}{4}}+d^{-3}$.

3. $m^{-\frac{5}{2}}-5m^{-2}n^2+10m^{-\frac{3}{2}}n^4-10m^{-1}n^6+5m^{-\frac{1}{2}}n^8-n^{10}$.

4. $x^3y^{-3}-9xy^{-1}+27x^{-1}y-27x^{-3}y^3$.

5. $x^{5m}+10x^{4m}y^n+40x^{3m}y^{2n}+80x^{2m}y^{3n}+80x^my^{4n}+32y^{5n}$.

6. $a^{12}+12a^9x^{\frac{1}{2}}+54a^6x+108a^3x^{\frac{3}{2}}+81x^2$.

7. $m^5n^{-5}-5m^{\frac{9}{2}}n^{-\frac{7}{2}}+10m^4n^{-2}-10m^{\frac{7}{2}}n^{-\frac{1}{2}}+5m^3n-m^{\frac{5}{2}}n^{\frac{5}{2}}$.

8. $xy^{-2}+3y^{-1}+3x^{-1}+x^{-2}y$.

9. $m^8-2m^6n^3+\frac{3}{2}m^4n^6-\frac{1}{2}m^2n^9+\frac{1}{16}n^{12}$.

10. $a^{\frac{5}{2}}b^{-\frac{5}{3}}-5a^{\frac{3}{2}}b^{-1}+10a^{\frac{1}{2}}b^{-\frac{1}{3}}-10a^{-\frac{1}{2}}b^{\frac{1}{3}}+5a^{-\frac{3}{2}}b-a^{-\frac{5}{2}}b^{\frac{5}{3}}$.

11. $a^6-12a^{\frac{29}{6}}+54a^{\frac{11}{3}}-108a^{\frac{5}{2}}+81a^{\frac{4}{3}}$.

12. $16x^4y^{-2}+16x^{\frac{5}{2}}y^{-\frac{1}{2}}+6xy+x^{-\frac{1}{2}}y^{\frac{5}{2}}+\frac{1}{16}x^{-2}y^4$.

13. $a^{-12}-2a^{-10}x^{\frac{1}{2}}+\frac{5}{3}a^{-8}x-\frac{20}{27}a^{-6}x^{\frac{3}{2}}+\frac{5}{27}a^{-4}x^2$
$-\frac{2}{81}a^{-2}x^{\frac{5}{2}}+\frac{1}{729}x^3$.

14. $x^3 + 15x^{\frac{12}{5}}y^{-\frac{2}{5}} + 90x^{\frac{9}{5}}y^{-\frac{4}{5}} + 270x^{\frac{6}{5}}y^{-\frac{6}{5}} + 405x^{\frac{3}{5}}y^{-\frac{8}{5}} + 243y^{-2}$.

15. $\frac{1}{8}a^3b^{\frac{3}{2}}x^{-\frac{3}{2}} - \frac{3}{2}b^{\frac{1}{2}}x^{-\frac{2}{3}} + 6a^{-3}b^{-\frac{1}{2}}x^{\frac{1}{6}} - 8a^{-6}b^{-\frac{3}{2}}x$.

16. $81a^{-3}b^2 - 108a^{-2}b + 54a^{-1} - 12b^{-1} + ab^{-2}$.

17. $a^3b^{-3} + 12a^2b^{-2} + 60ab^{-1} + 160 + 240a^{-1}b$
$+ 192a^{-2}b^2 + 64a^{-3}b^3$.

18. $1 - 4x + 2x^2 + 8x^3 - 5x^4 - 8x^5 + 2x^6 + 4x^7 + x^8$.

19. $x^8 + 4x^7 - 2x^6 - 20x^5 + x^4 + 40x^3 - 8x^2 - 32x + 16$.

20. $1 + 8x + 20x^2 + 8x^3 - 26x^4 - 8x^5 + 20x^6 - 8x^7 + x^8$.

21. $1 - 5x + 15x^2 - 30x^3 + 45x^4 - 51x^5 + 45x^6 - 30x^7$
$+ 15x^8 - 5x^9 + x^{10}$.

Art. 367; page 305.

2. $462a^5x^6$.
3. $252m^5$.
4. $-792c^5d^7$.
5. $1001a^8$.
6. $84a^{-3}b^3$.
7. $715x^{11}$.
8. $-\frac{63}{16}a^{-7}b^5$.
9. $15360x^{-3}y^{\frac{7}{2}}$.
10. $-4455a^{-\frac{16}{3}}x^{-3}$.
11. 126720.

Art. 378; pages 311, 312.

2. $1 - 2x + 2x^2 - 2x^3 + 2x^4 - \cdots$
3. $2 + 11x + 33x^2 + 99x^3 + 297x^4 + \cdots$
4. $3 - 19x^2 + 95x^4 - 475x^6 + 2375x^8 - \cdots$
5. $\frac{2}{3}x + \frac{4}{9}x^3 + \frac{8}{27}x^5 + \frac{16}{81}x^7 + \frac{32}{243}x^9 + \cdots$
6. $1 - 2x + 2x^3 - 2x^4 + 2x^6 - \cdots$
7. $x - x^2 - 2x^3 - 5x^4 - 12x^5 - \cdots$
8. $2 - x + 3x^2 - x^3 + 3x^4 - \cdots$
9. $1 - 2x + 5x^2 - 16x^3 + 47x^4 - \cdots$
10. $2 - 7x + 28x^2 - 91x^3 + 322x^4 - \cdots$

11. $\frac{1}{2}x^2+\frac{5}{4}x^3+\frac{7}{8}x^4+\frac{17}{16}x^5+\frac{31}{32}x^6+\cdots$

12. $1+\frac{1}{3}x+\frac{1}{3}x^2-\frac{8}{9}x^3-\frac{11}{27}x^5+\cdots$

13. $\frac{1}{2}+\frac{3}{4}x-\frac{3}{8}x^2-\frac{1}{16}x^3+\frac{21}{32}x^4+\cdots$

15. $\frac{2}{3}x^{-2}+\frac{8}{9}x^{-1}+\frac{32}{27}+\frac{128}{81}x+\frac{512}{243}x^2+\cdots$

16. $x^{-1}+3+2x-5x^2-16x^3-\cdots$

17. $x^{-2}-x^{-1}-2x+2x^2-4x^3+\cdots$

18. $\frac{3}{2}x^{-3}-\frac{1}{4}x^{-2}-\frac{1}{8}x^{-1}+\frac{31}{16}+\frac{23}{32}x+\cdots$

Art. 379; page 314.

2. $1+x-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{8}x^4+\cdots$

3. $1-\frac{3}{2}x-\frac{9}{8}x^2-\frac{27}{16}x^3-\frac{405}{128}x^4-\cdots$

4. $1-x+x^2+x^3+\frac{1}{2}x^4+\cdots$

5. $1+\frac{1}{2}x-\frac{5}{8}x^2+\frac{5}{16}x^3-\frac{45}{128}x^4+\cdots$

6. $1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3-\frac{10}{243}x^4-\cdots$

7. $1+\frac{1}{3}x+\frac{2}{9}x^2-\frac{13}{81}x^3+\frac{8}{243}x^4-\cdots$

Art. 381; page 316.

3. $\frac{6}{2x+5}+\frac{1}{2x-5}.$

4. $\frac{3}{x}-\frac{5}{3x+5}.$

5. $\frac{5}{2x}-\frac{1}{x+3}-\frac{1}{x-3}.$

6. $-\frac{2}{x}+\frac{3}{3x+1}+\frac{2}{2x-5}.$

7. $\frac{3a}{x+a}-\frac{2a}{x-4a}.$

8. $\frac{1}{3+4x}+\frac{2}{3-x}.$

9. $\dfrac{1}{x+1}+\dfrac{2}{2x+3}-\dfrac{3}{2x-3}$.

10. $\dfrac{1}{x+2}-\dfrac{1}{x-2}-\dfrac{3}{x+1}+\dfrac{3}{x-1}$.

11. $\dfrac{1+\sqrt{2}}{2x-5+\sqrt{2}}+\dfrac{1-\sqrt{2}}{2x-5-\sqrt{2}}$.

Art. 383; page 317.

2. $\dfrac{2}{x+5}-\dfrac{23}{(x+5)^2}$. 3. $\dfrac{1}{x-2}+\dfrac{4}{(x-2)^2}+\dfrac{4}{(x-2)^3}$.

4. $\dfrac{3}{x+1}-\dfrac{6}{(x+1)^2}-\dfrac{1}{(x+1)^3}$.

5. $\dfrac{2}{3x+2}-\dfrac{4}{(3x+2)^2}-\dfrac{3}{(3x+2)^3}$.

6. $\dfrac{1}{5(5x-2)}-\dfrac{4}{5(5x-2)^3}$.

7. $\dfrac{1}{x+1}+\dfrac{1}{(x+1)^2}-\dfrac{1}{(x+1)^3}-\dfrac{1}{(x+1)^4}$.

8. $\dfrac{2}{x-1}-\dfrac{4}{(x-1)^2}+\dfrac{3}{(x-1)^3}-\dfrac{1}{(x-1)^4}$.

9. $\dfrac{1}{2(2x-3)}-\dfrac{27}{2(2x-3)^3}-\dfrac{27}{(2x-3)^4}$.

Art. 384; page 319.

2. $\dfrac{2}{x}-\dfrac{3}{x+2}-\dfrac{5}{(x+2)^2}$. 3. $\dfrac{5}{x}-\dfrac{1}{x^2}-\dfrac{5}{x+1}-\dfrac{4}{(x+1)^2}$.

4. $\dfrac{1}{x-2}-\dfrac{1}{2(2x-3)}+\dfrac{3}{2(2x-3)^2}$.

5. $\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x-2}+\dfrac{1}{(x-2)^2}$.

6. $\dfrac{1}{x}-\dfrac{2}{x^2}+\dfrac{3}{x^3}-\dfrac{4}{x+5}$.

7. $\dfrac{5}{x}-\dfrac{1}{x^2}+\dfrac{2}{x^3}-\dfrac{5}{x+1}-\dfrac{4}{(x+1)^2}$.

Art. 385; page 320.

1. $2x-5-\dfrac{17}{2x-5}+\dfrac{2}{2x+1}.$

2. $3+\dfrac{1}{x+2}-\dfrac{5}{(x+2)^2}-\dfrac{18}{(x+2)^3}.$

3. $5x^2+\dfrac{1}{x}-\dfrac{3}{x^2}+\dfrac{3}{x^3}-\dfrac{1}{x+1}.$

4. $3x-2-\dfrac{1}{x+1}+\dfrac{2}{(x+1)^2}+\dfrac{7}{x-1}+\dfrac{8}{(x-1)^2}.$

5. $2x^2-7-\dfrac{2}{x}+\dfrac{1}{x^3}-\dfrac{5}{x-1}.$

Art. 387; page 322.

1. $x=y-y^2+y^3-y^4+\cdots$

2. $x=\dfrac{1}{3}y+\dfrac{2}{27}y^2-\dfrac{1}{243}y^3-\dfrac{14}{2187}y^4+\cdots$

3. $x=2y+6y^2+\dfrac{68}{3}y^3+98y^4+\cdots$

4. $x=(y-1)-\dfrac{1}{2}(y-1)^2+\dfrac{1}{3}(y-1)^3-\dfrac{1}{4}(y-1)^4+\cdots$

5. $x=y+y^3+2y^5+5y^7+\cdots$

6. $x=2y+\dfrac{8}{3}y^2+\dfrac{28}{9}y^3+\dfrac{464}{135}y^4+\cdots$

7. $x=\dfrac{1}{3}y-\dfrac{5}{27}y^2+\dfrac{29}{243}y^3-\dfrac{199}{2187}y^4+\cdots$

8. $x=y+\dfrac{1}{3}y^3+\dfrac{2}{15}y^5+\dfrac{17}{315}y^7+\cdots$

Art. 392; page 327.

7. $a^{\frac{1}{2}}-\dfrac{1}{2}a^{-\frac{1}{2}}x-\dfrac{1}{8}a^{-\frac{3}{2}}x^2-\dfrac{1}{16}a^{-\frac{5}{2}}x^3-\dfrac{5}{128}a^{-\frac{7}{2}}x^4-\cdots$

8. $1-\dfrac{1}{3}x+\dfrac{2}{9}x^2-\dfrac{14}{81}x^3+\dfrac{35}{243}x^4-\cdots$

9. $a^{-3}+3a^{-4}x+6a^{-5}x^2+10a^{-6}x^3+15a^{-7}x^4+\cdots$

10. $c^{-\frac{3}{2}}-c^{-3}d+c^{-\frac{9}{2}}d^2-c^{-6}d^3+c^{-\frac{15}{2}}d^4-\cdots$

11. $x^{-\frac{1}{3}}-2x^{\frac{1}{6}}y-x^{\frac{2}{3}}y^2-\frac{4}{3}x^{\frac{7}{6}}y^3-\frac{7}{3}x^{\frac{5}{3}}y^4-\cdots$

12. $a^{-\frac{1}{4}}+\frac{1}{2}a^{-\frac{5}{4}}x^2+\frac{5}{8}a^{-\frac{9}{4}}x^4+\frac{15}{16}a^{-\frac{13}{4}}x^6+\frac{195}{128}a^{-\frac{17}{4}}x^8+\cdots$

13. $a^{-\frac{5}{3}}-a^{-\frac{5}{2}}b^{-1}+\frac{3}{4}a^{-\frac{10}{3}}b^{-2}-\frac{1}{2}a^{-\frac{25}{6}}b^{-3}+\frac{5}{16}a^{-5}b^{-4}-\cdots$

14. $x^3+3x^{-1}ab-\frac{3}{2}x^{-5}a^2b^2+\frac{5}{2}x^{-9}a^3b^3-\frac{45}{8}x^{-13}a^4b^4+\cdots$

15. $1-10xy^{-1}+80x^2y^{-2}-\frac{1760}{3}x^3y^{-3}+\frac{12320}{3}x^4y^{-4}-\cdots$

16. $a^4+12a^5y^{-2}+90a^6y^{-4}+540a^7y^{-6}+2835a^8y^{-8}+\cdots$

17. $128a^7+112a^5x^{-\frac{2}{3}}+35a^3x^{-\frac{4}{3}}+\frac{35}{8}ax^{-2}+\frac{35}{256}a^{-1}x^{-\frac{8}{3}}-\cdots$

18. $m+3m^{\frac{5}{3}}n^{\frac{3}{2}}+\frac{15}{2}m^{\frac{7}{3}}n^3+\frac{35}{2}m^3n^{\frac{9}{2}}+\frac{315}{8}m^{\frac{11}{3}}n^6+\cdots$

Art. 393; page 328.

2. $\frac{33}{2048}a^{-\frac{13}{2}}x^7.$

3. $-364m^{11}.$

4. $\frac{315}{128}a^8.$

5. $-\frac{5}{1024}a^{-\frac{7}{2}}x^6.$

6. $-\frac{9009}{256}a^{-34}b^{\frac{5}{2}}.$

7. $\frac{44}{6561}x^{\frac{14}{3}}y^{-8}.$

8. $-\frac{663}{8192}x^{-\frac{21}{2}}y^{15}.$

9. $\frac{77}{256}a^{-\frac{13}{4}}x^{10}.$

10. $3003n^{10}c^{-16}.$

11. $-\frac{308}{3}a^{-\frac{34}{9}}x^{-5}.$

12. $36x^{-30}y^{-10}z^{-\frac{14}{3}}.$

Art. 394; page 329.

2. 3.16228.

3. 9.94988.

4. 2.08008.

5. 2.97182.

6. 2.03055

7. 1.94729

Art. 407; page 333.

2. 1.3222.	**7.** 1.9912.	**12.** 2.1303.	**17.** 3.0545.
3. 1.7993.	**8.** 2.0212.	**13.** 2.2252.	**18.** 3.7114.
4. 1.7481.	**9.** 2.0491.	**14.** 2.1673.	**19.** 3.8484.
5. 1.9242.	**10.** 2.1582.	**15.** 2.5741.	**20.** 4.1585.
6. 1.6532.	**11.** 2.3343.	**16.** 2.5353.	**21.** 4.1915.

Art. 409; page 334.

2. .3680.	**5.** 1.5441.	**8.** .2252.	**11.** .8539.
3. .1549.	**6.** .1182.	**9.** 2.2431.	**12.** .7660.
4. .5229.	**7.** 2.0970.	**10.** 1.0458.	**13.** .7360.

Art. 412; page 335.

3. .2863.	**9.** 4.5844.	**15.** .1165.	**22.** .2601.
4. 2.7090.	**10.** 3.2620.	**16.** .3860.	**23.** .6884.
5. 4.2255.	**11.** .9801.	**17.** .2212.	**24.** .1840.
6. .1398.	**12.** .4225.	**18.** .1750.	**25.** .2215.
7. .7194.	**13.** .1590.	**20.** 2.6145.	**26.** .2494.
8. .6611.	**14.** .0430.	**21.** .1678.	**27.** .1449.

Art. 414; page 337.

2. .2552.	**7.** 7.7323 – 10.	**12.** 2.4804.
3. .3522.	**8.** 6.4983 – 10.	**13.** 8.7905 – 10.
4. 9.2922 – 10.	**9.** 3.8663.	**14.** 6.3588.
5. 8.6811 – 10.	**10.** .6074.	**15.** .1964.
6. 1.5841.	**11.** 9.6511 – 10.	**16.** .1688.

Art. 420; page 341.

7. 9.8878 – 10.	**11.** 1.3028.	**15.** 0.7144.
8. 3.0237.	**12.** 4.9659.	**16.** 3.0155.
9. 0.5177.	**13.** 9.6055 – 10.	**17.** 8.9379 – 10.
10. 8.7164 – 10.	**14.** 7.8560 – 10.	**18.** 9.0610 – 10.

Art. 421; page 343.

6. 1.646.	**10.** .003318.	**13.** .2079.	**16.** 63329.
7. 8886.	**11.** 10221.	**14.** 44.48.	**17.** .01301.
9. .01461.	**12.** 9.492.	**15.** .001109.	**18.** 502.9.

Art. 426; pages 346 to 348.

1. 8.454.	**19.** -1.184.	**39.** .6443.
2. 10.73.	**20.** .000007038.	**40.** .5010.
3. -2202.	**21.** 2.924.	**41.** 1.062.
4. .2179.	**22.** .9146.	**42.** $-.9102$.
5. .01157.	**23.** 4.638.	**43.** 1.093.
6. $-.7032$.	**24.** .0000639.	**44.** .7035.
7. 7.672.	**25.** 1.414.	**45.** .5807.
8. .6688.	**26.** 1.495.	**46.** $-.6313$.
9. -3.908.	**27.** -1.246.	**47.** 24.62.
10. 1782.	**28.** .6553.	**48.** .2979.
11. .3500.	**29.** .2846.	**49.** 98.50.
12. $-.4748$.	**30.** 2.372.	**50.** 1.660.
13. .4127.	**31.** $-.5142$.	**51.** 3.076.
14. -4.671.	**32.** .1588.	**52.** .8678.
15. .2415.	**35.** 5.883.	**53.** 1.134.
16. $-.0725$.	**36.** .7885.	**54.** .5881.
17. 13587.	**37.** 1.195.	**55.** 1.805.
18. .006415.	**38.** .6803.	**56.** .003229.
		57. .03344.

Art. 427; page 349.

3. .4581. **4.** .1853. **5.** $-.4949$. **6.** $-.2601$.

7. $\dfrac{m\log b + n\log c}{\log a}$. **9.** 3, -1. **10.** -3.

8. $\dfrac{\log a}{\log n - \log m}$. **11.** $n = \dfrac{\log l - \log a}{\log r} + 1$.

12. $n = \dfrac{\log[(r-1)S+a] - \log a}{\log r}.$

13. $n = \dfrac{\log l - \log a}{\log(S-a) - \log(S-l)} + 1.$

14. $n = \dfrac{\log l - \log[rl - (r-1)S]}{\log r} + 1.$

16. 3.4598. 18. -3.467. 20. .9395.

17. $-.1386$. 19. 11.193. 21. -1.8204.

Art. 437; page 353.

3. 4.479. 4. 7.19. 5. -1.07. 6. -2.4576.

Art. 439; pages 356, 357.

1. \$2853.75. 2. \$702.86. 3. $5\frac{1}{2}$. 4. 4.

5. 14.198. 6. 16.01. 7. \$647.14.

Art. 443; page 359.

1. \$2076.40. 3. \$2959.18. 4. \$277. 5. \$576.50.

Art. 452; pages 363 to 365.

3. 7893600. 5. 126. 7. 3838380.

4. 5040. 6. 15120. 8. 31824.

9. 360; 120; 720; 1956.

10. 134596. 14. 15840. 17. 10584000.

11. 125970. 15. 121030. 18. 3303300.

12. 4533. 16. 10080. 19. 720.

Art. 466; pages 375, 376.

1. $1 + \dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\,\dfrac{1}{2}$; 5th convergent, $\dfrac{19}{14}$.

2. $\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\,\dfrac{1}{3}$; 5th convergent, $\dfrac{15}{19}$.

3. $3+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{2}$; 5th convergent, $\frac{18}{5}$.

4. $\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2}$; 5th convergent, $\frac{8}{11}$.

5. $2+\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2}$; 5th convergent, $\frac{85}{37}$.

6. $1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}$; 5th convergent, $\frac{8}{5}$.

7. $1+\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3}$; 5th convergent, $\frac{24}{19}$.

8. $\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5+}\,\frac{1}{6+}\,\frac{1}{7}$; 5th convergent, $\frac{68}{157}$.

9. $2+\frac{1}{4+}\,\frac{1}{4+\cdots}$; 4th convergent, $\frac{161}{72}$; $\frac{1}{27144}$, $\frac{1}{21960}$.

10. $1+\frac{1}{1+}\,\frac{1}{2+\cdots}$; 4th convergent, $\frac{7}{4}$; $\frac{1}{60}$, $\frac{1}{44}$.

11. $3+\frac{1}{3+}\,\frac{1}{6+\cdots}$; 4th convergent, $\frac{199}{60}$; $\frac{1}{26340}$, $\frac{1}{22740}$.

12. $2+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+\cdots}$; 4th convergent, $\frac{8}{3}$; $\frac{1}{51}$, $\frac{1}{42}$.

13. $\frac{-3+\sqrt{15}}{2}$.

14. $-2+2\sqrt{2}$.

15. $\frac{3+\sqrt{5}}{2}$.

16. $-1+2\sqrt{6}$.

17. $3+\frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+\cdots}$; 4th convergent, $\frac{355}{113}$.

18. $\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{7+\cdots}$; $\frac{76}{175}$; $\frac{1}{262325}$, $\frac{1}{231700}$.

19. $2+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{19+\cdots}$; $\frac{193}{71}$;
$\frac{1}{103589}$, $\frac{1}{98548}$.

20. $3+\frac{1}{1+}\,\frac{1}{5+\cdots}$; 5th convergent, $\frac{158}{41}$.

Art. 475; page 380.

2. $4, -7.$ **3.** $\frac{9}{2}, -5.$ **4.** $\frac{3}{4}, \frac{2}{3}.$ **5.** $\frac{5 \pm 2\sqrt{3}}{4}.$ **6.** $-\frac{5}{2}, -\frac{5}{3}.$

7. $1, \frac{3}{5}.$ **8.** $-\frac{1}{5}.$ **9.** $\frac{7}{2}, -\frac{1}{3}.$ **10.** $-2a, -4a.$ **11.** $m+3, -m-2.$

Art. 476; page 381.

2. $x^3 - 6x^2 + 11x - 6 = 0.$ **4.** $x^4 - 21x^2 + 20x = 0.$

3. $x^3 - 19x - 30 = 0.$ **5.** $24x^3 + 46x^2 + 29x + 6 = 0.$

6. $6x^4 - 37x^3 - 4x^2 + 57x + 18 = 0.$

7. $9x^4 + 6x^3 - 59x^2 - 20x + 100 = 0.$

8. $12x^4 - 13x^3 - 144x^2 + 13x + 12 = 0.$

9. $x^4 - 14x^2 + 1 = 0.$

10. $16x^4 + 16x^3 - 112x^2 - 148x + 19 = 0.$

Art. 479; page 384.

8. $-2, -3.$ **9.** $3, -5.$

Art. 486; page 388.

2. $x^3 - 15x^2 - 63x + 297 = 0.$

3. $x^4 - 30x^3 + 250x - 3125 = 0.$

4. $27x^3 - 30x + 28 = 0.$

5. $96x^4 + 60x^3 + 200x^2 - 625 = 0.$

Art. 487; page 389.

2. $x^3 + 6x^2 + 2x - 64 = 0.$

3. $x^3 - 40x + 650 = 0.$

4. $x^3 + 6x^2 - 735 = 0.$

5. $x^4 + 30x^3 - 48x^2 - 1458 = 0.$

6. $x^4 - 12x^3 + 45x - 243 = 0.$

7. $x^4 - 35x^2 - 90x + 270 = 0.$

Art. 491; page 391.

2. $x^3 + 14x^2 + 57x - 4 = 0.$

3. $x^3 - 20x^2 + 129x - 293 = 0.$

4. $x^4 - 13x^3 + 61x^2 - 116x - 12 = 0.$

5. $x^4 + 24x^3 + 219x^2 + 895x + 1376 = 0.$

Art. 492; page 392.

2. $y^3 - 30y + 68 = 0.$ 3. $y^3 - \frac{1}{3}y - \frac{110}{27} = 0.$

4. $y^4 - 24y^2 - 69y - 57 = 0.$

5. $y^5 - 10y^3 + 11y^2 + 3y - 33 = 0.$

Art. 497; page 395.

5. Two positive, two negative.
7. Two positive, three negative.
8. Three positive, two negative.
9. One positive, two imaginary.
10. One positive, two imaginary.
11. Four imaginary.
12. One negative, four imaginary.
13. One positive, four imaginary.
14. One negative, four imaginary.

Art. 498; page 396.

2. Second, 4. 3. Third, 6. 4. Fourth, 72. 5. Fourth, 24. 6. Fifth, 240. 7. Fifth, 600.

Art. 501; page 398.

2. $2, 2, -7.$ 4. $\frac{3}{2}, \frac{3}{2}, -\frac{7}{2}.$ 6. $-3, -3, -3, 2.$

3. $-1, -1, 6.$ 5. $1, 1, -4, -4.$ 7. $1, 1, -2, -2, 2.$

8. $6, 6, -2, -4.$

Art. 503; page 401.

2. One each between 0 and 1, 4 and 5, 0 and -1.

3. One each between 1 and 2, 4 and 5, 0 and -1.

4. One each between 0 and 1, 0 and -1, -2 and -3.

5. One each between 1 and 2, 2 and 3, -1 and -2, -2 and -3.

6. One each between 1 and 2, 0 and -1, -8 and -9.

7. One each between 1 and 2, 3 and 4, 0 and -1, -3 and -4.

8. One each between 2 and 3, -1 and -2, -3 and -4, -4 and -5.

9. One each between 0 and 1, 1 and 2, 4 and 5, -1 and -2.

Art. 510; page 408.

3. One each between 1 and 2, 4 and 5, -1 and -2.

4. One between 0 and -1; two imaginary roots.

5. One each between 0 and 1, 1 and 2, -1 and -2.

6. One each between 2 and 3, 0 and -1, -4 and -5.

7. Two between 3 and 4, one between -3 and -4.

8. Two between 0 and 1; one each between 2 and 3, -3 and -4.

9. One each between 0 and 1, -1 and -2; two imaginary roots.

10. One each between 0 and 1, 1 and 2; two between -2 and -3.

Art. 514; pages 413, 414.

3. 1, 3, 4.

4. -1, 6, -5.

5. -2, $\dfrac{-3 \pm \sqrt{57}}{2}$.

6. 1, $\frac{5}{2}$, -4.

7. 2, 8, -3.

8. $-\frac{2}{3}$, ± 1.

9. $1, 2, -2, -3.$

10. $1, -1, 3, -2.$

11. $-1, 2, -2, -5.$

12. $\frac{1}{2}, \frac{3}{2}, 2, -1.$

13. $-1, -2, -3, -5.$

14. $1, 2, 3, -7.$

15. $-\frac{1}{2}, \frac{3}{2}, 2, -3.$

16. $-1, 3, 4, 5.$

17. $\frac{4}{3}, -1, \frac{-1 \pm \sqrt{13}}{6}.$

18. $-1, 3, \frac{5 \pm \sqrt{-7}}{2}.$

Art. 523; pages 418, 419.

2. $-1, \frac{3}{2}, \frac{2}{3}.$

3. $1, -3 \pm 2\sqrt{2}.$

4. $\pm 1, -5, -\frac{1}{5}.$

5. $1, \frac{a-1 \pm \sqrt{a^2-2a-3}}{2}.$

6. $3, \frac{1}{3}, -\frac{3}{5}, -\frac{5}{3}.$

7. $1, -3 \pm 2\sqrt{2}, \frac{5 \pm \sqrt{21}}{2}.$

8. $-1, \frac{-7 \pm 3\sqrt{5}}{2}, \frac{1 \pm \sqrt{-3}}{2}.$

9. $\pm 1, \pm\sqrt{-1}, \frac{1 \pm \sqrt{-3}}{2}.$

10. $1, 2, \frac{1}{2}, -\frac{3}{4}, -\frac{4}{3}.$

11. $-1, -3\ -\frac{1}{3}, \frac{-1 \pm \sqrt{-3}}{2}.$

Art. 524; page 419.

1. $1, \frac{1}{4}(-1 + \sqrt{5} \pm \sqrt{-10 - 2\sqrt{5}}),$
$\frac{1}{4}(-1 - \sqrt{5} \pm \sqrt{-10 + 2\sqrt{5}}).$

2. $-1, \frac{1}{4}(1 + \sqrt{5} \pm \sqrt{-10 + 2\sqrt{5}}),$
$\frac{1}{4}(1 - \sqrt{5} \pm \sqrt{-10 - 2\sqrt{5}}).$

3. $2, \frac{1}{2}(-1 + \sqrt{5} \pm \sqrt{-10 - 2\sqrt{5}}),$
$\frac{1}{2}(-1 - \sqrt{5} \pm \sqrt{-10 + 2\sqrt{5}}).$

Art. 529; page 421.

2. $-4, 2 \pm 3\sqrt{-3}.$

3. $6, -3, -3.$

4. $2, -1 \pm 6\sqrt{-3}.$

5. $10, -2 \pm \sqrt{-3}.$

6. $-1, 5 \pm 4\sqrt{-3}$.

7. $-7, 3, 3$.

8. $-2, -5 \pm 2\sqrt{-3}$.

9. $6, -1, -1$.

10. $-1, \frac{1}{2}(3 \pm \sqrt{-3})$.

11. $4, \dfrac{-5 \pm 3\sqrt{-3}}{2}$.

12. $\sqrt[3]{\left(\dfrac{-9+\sqrt{69}}{18}\right)} + \sqrt[3]{\left(\dfrac{-9-\sqrt{69}}{18}\right)}$.

Art. 538; pages 426, 427.

1. 1.2016, $-$1.33005.

2. 5.1345.

3. $-$2.1768.

4. .0945.

5. $-$5.7683.

6. 3.23606.

7. $-$2.1574.

8. .6457.

9. 2.2469, .5549, $-$.8019.

10. 1.8793, $-$.3472, $-$1.53208.

11. $-$2.2134.

12. 2.0472, .5936.

13. 1.3085, $-$1.1365.

14. 2.3568, 2.69202, $-$2.0489.

15. 1.2599.

16. 2.5712.

17. 1.4953.